Proving Equivalence

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Figure 1: Proof of $(A \wedge B) \rightarrow (B \wedge A)$.

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- We'd like to have our logic be capable of deriving the fact that $\varphi \wedge \psi$ and $\psi \wedge \varphi$ are *equivalent* statements.
- Remembering that we already have a way to state equivalence in our logic via the biimplicational connective $\leftrightarrow$, we add more logical rules.

Introducing $\leftrightarrow$

Inference Rule 9 (Biimplication Introduction)

$$\frac{\Gamma \vdash \varphi \to \psi \qquad \Delta \vdash \psi \to \varphi}{\Gamma, \Delta \vdash \varphi \leftrightarrow \psi}\,(\leftrightarrow\mathrm{I})$$

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- There are also elimination rules for $\leftrightarrow$ that let us use equivalences in proofs.

Eliminating $\leftrightarrow$

Inference Rule 10 (Biimplication Elimination 1)

$$\frac{\Gamma \vdash \varphi \leftrightarrow \psi}{\Gamma \vdash \varphi \rightarrow \psi} \; ({\leftrightarrow}\mathrm{E}_1)$$

Inference Rule 11 (Biimplication Elimination 2)

$$\frac{\Gamma \vdash \varphi \leftrightarrow \psi}{\Gamma \vdash \psi \rightarrow \varphi} \; ({\leftrightarrow}\mathrm{E}_2)$$

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Figure 2: Proof of $(A \wedge B) \leftrightarrow (B \wedge A)$.

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- This means that what we've proved, namely that $A \wedge B$ and $B \wedge A$ are equivalent to one another, is not contingent on any other assumptions. This is exactly what we want our logic to say about equivalences.
- One technical note: I have used A and B in the proof in Figure 2. But a similiar proof would work for any two propositions, not just atomic ones.

Exercises

Problem 1

Starting with the assumptions $A \leftrightarrow B$, $(B \wedge A) \rightarrow C$, and A, give a sequent-style natural deduction proof of $A \rightarrow C$.