Using Natural Deduction to Represent Arguments (Part 2)

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Example Argument

Consider the following:

(1)
a. Pastor Ingqvist and Father Wilmer go fishing.
b. If Pastor Ingqvist goes Fishing, no one receives the lutefisk shipment.
c. If no one receives the lutefisk shipment and today is Saturday, the festival is canceled.
d. Today is Saturday.
e. That means the festival must be canceled.

Analyzing the Example

To analyze the argument in (1) for validity, we proceed as usual, starting by identifying the atomic propositions it uses.

P Pastor Ingqvist goes fishing.

W Father Wilmer goes fishing.

L Someone receives the lutefisk shipment.

S Today is Saturday.

C The festival is canceled.

Translating the Argument into PL

Now we can translate the declarative sentences used to construct the argument in (1) into propositions:

$$P \wedge W \tag{1a}$$

$$P \rightarrow \neg L \tag{1b}$$

$$(\neg L \wedge S) \rightarrow C \tag{1c}$$

$$S \tag{1d}$$

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So as usual, we have an argument with some premises (1a-1d) and a conclusion (1e).

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- But the inference step we'll use to go from $P \to \neg L$ to $\neg L$ given P will only give us a proof of $\neg L$ by itself.
- We know, both intuitively and via truth table verification that A being true and B being true means that $A \wedge B$ is also true.
- And while we can sometimes just add more assumptions containing just the left and right sides of a conjunction, this won't work all the time.
- We need more rules to handle this argument using natural deduction.

Rule for Introducing $\wedge$

As before, φ and ψ are meta-variables ranging over propositions (atomic or complex).

Inference Rule 4 (Conjunction Introduction)

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\ (\wedge\mathrm{I})$$

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- Rule 4 is both intuitively straightforward and easily verifiable using a truth table.

Rules for Eliminating ∧

To "unpack" a conjunction into its component parts, we need two rules that essentially do the same thing:

Inference Rule 5 (Conjunction Elimination 1)

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Inference Rule 6 (Conjunction Elimination 2)

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- Rules 5 and 6 are mirror images of each other.
- They say that if you’ve proved the conjunction $\varphi \wedge \psi$ then you can deduce that you’ve proved either of the conjuncts.

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- So, given that we already have a way to eliminate the $\rightarrow$ connective, Figure 1 contains a formal proof of the argument in (1).

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \wedge W}\,(\text{Hyp})}{P}\,(\wedge \text{E}_1) \qquad \dfrac{}{P \rightarrow \neg L}\,(\text{Hyp})}{\neg L}\,(\rightarrow\text{E}) \qquad \dfrac{}{S}\,(\text{Hyp})}{\neg L \wedge S}\,(\wedge\text{I}) \qquad \dfrac{}{(\neg L \wedge S) \rightarrow C}\,(\text{Hyp})$$

$$\dfrac{\neg L \wedge S \qquad (\neg L \wedge S) \rightarrow C}{C}\,(\rightarrow\text{E})$$

Figure 1: Proof of the argument in (1).

Exercises

Problem 1

We know, both intuitively and from truth tables, that for any two propositions φ and ψ the propositions $\varphi \land \psi$ and $\psi \land \varphi$ are equivalent. Give a formal proof that has $A \land B$ as its premise and $B \land A$ as its conclusion. That is, you should come up with a proof tree that looks like

$$\dfrac{}{A \land B}\,(\text{Hyp})$$

$$\dfrac{\vdots}{B \land A}\,(?)$$

where you fill in the $\vdots$ and ?s. (Hint: you will use the rules for $\land$ talked about above.)