# Entailments and Equivalence

Scott Martin

February 20, 2012

## Taking Stock

### Graphing Arguments

- With natural deduction (ND), we can now draw graphs of arguments and examine them for validity without having to use truth tables.
- But we'd also like to be able to discover equivalences between propositions expressed in propositional logic (PL).
- Also, we'd like to have a way to say (formally) when some premise(s) entail some conclusion.

## Proofs of Equivalence

### Proving What We've Known All Along

- Since we started talking about $\wedge$ (logical conjunction) and how it conjoins propositions, we've noted that for any propositions $\varphi$ and $\psi$ the proposition $\varphi \wedge \psi$ is equivalent to the proposition $\psi \wedge \varphi$.
- We've said things like "it doesn't matter which side" or "they're interchangeable" to indicate this is our informal discussions.
- Notice that you *could* use truth tables to quickly convince yourself that $\varphi \wedge \psi$ is true in all the same cases that $\psi \wedge \varphi$ is true.
- But with ND, we don't need to do everything by interpretation anymore. We can give a **syntactic** proof (shown in Figure 1) that says the same thing without needing to consider every way the world could be.

$$\dfrac{\dfrac{\dfrac{}{A \wedge B}\,(\text{Hyp})}{B}\,(\wedge\text{E}_2) \qquad \dfrac{\dfrac{}{A \wedge B}\,(\text{Hyp})}{A}\,(\wedge\text{E}_1)}{B \wedge A}\,(\wedge\text{I})$$

Figure 1: Proof of $B \wedge A$ from $A \wedge B$.

# ND Proofs and Entailment

### Graphical Entailment

- In ND proofs, we can see graphically how an argument is laid out: the premises are at the top, the conclusion at the very bottom, and everything in between is an inference step allowed by one of our rules.
- Notice how this mirrors the notion of entailment for deductively valid arguments–starting with true premises, you arrive at a conclusion that must be true (no matter how the world is).
- We can mention this in the meta-language, but sometimes we'd actually like to use the fact that some premise(s) entail some conclusion in yet another proof.
- One way to do this is to simply re-use the conclusion as a new premise (we'll do this a lot).

## Implication Introduction

### Introducing Implication

- Another way is to make an implication using one of the premises as the antecedent and the conclusion as the consequent (Rule 7 formalizes this step).

**Inference Rule 7** (Implication Introduction)**.**

$$\cfrac{}{[\varphi]_i}\ \text{(Hyp)}$$

$$\vdots$$

$$i\ \cfrac{\psi}{\varphi \to \psi}\ (\to\text{I})$$

### Rule 7 Explained

- Rule 7 looks complicated (we'll fix this later), but all it says is that if you've got some premise that you've assumed using Hypothesis, you can later **withdraw** or **revoke** that hypothesis and make it the antecedent of a conditional.
- For bookkeeping, we pick an as-yet-unused number $i$ (it doesn't matter which) to label the introduction step. We then write $[\quad]_i$ around the withdrawn hypothesis, and the same number $i$ to the left of the introduction step.
- Notice that our rules are getting a bit sloppy–now we have to know that $\vdots$ means something like "any number of inference steps".
- So the rule of Implication Introduction is not as formally rigorous as some of the other rules we've used up to this point because it relies more on the meta-language.

### Example of Rule 7 In Action

- Figure 2 shows an example of this rule in action.
- Notice that the choice to start numbering at 1 gives us an easy mnemonic to remember the order the introductions occurred in.
- Also note that the $[\quad]_i$ makes it clear that the withdrawn premises were only assumed "for the time being" and are no longer required assumptions for the truth of the final conclusion.

$$
\dfrac{\dfrac{}{[A \to B]_2}\ (\text{Hyp}) \qquad \dfrac{\dfrac{}{[A \wedge C]_1}\ (\text{Hyp})}{A}\ (\wedge\text{E}_1)}{2\ \dfrac{1\ \dfrac{B}{(A \wedge C) \to B}\ (\to\text{I})}{(A \to B) \to ((A \wedge C) \to B)}\ (\to\text{I})}\ (\to\text{E})
$$

Figure 2: Example proof using Rule 7.

# Homework

### Exercises

**Problem 1.** Give a formal ND proof of $\neg\neg A \to B$ that assumes as premises only $\neg\neg A$ and $A \to B$.