Proving Equivalence

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Equivalences and Implication

Capturing Entailment

- We saw recently that it's always possible (for any two PL propositions φ and ψ) to put either one on either side of a conjunction.
- Another way of saying this is that any time $\varphi \wedge \psi$ is true $\psi \wedge \varphi$ is also true (as our truth tables can verify).
- So we can start from either $\varphi \wedge \psi$ or $\psi \wedge \varphi$ and prove the other.
- Also, now that we have Implication Introduction ($\to$I), we can capture a piece of the entailment present in any given proof (Figure 1 shows an example of this).

$$\dfrac{\dfrac{\dfrac{A \wedge B \vdash A \wedge B}{A \wedge B \vdash B}\,(\wedge \mathrm{E}_2) \qquad \dfrac{A \wedge B \vdash A \wedge B}{A \wedge B \vdash A}\,(\wedge \mathrm{E}_1)}{A \wedge B \vdash B \wedge A}\,(\wedge \mathrm{I})}{\vdash (A \wedge B) \to (B \wedge A)}\,(\to \mathrm{I})$$

Figure 1: Proof of $(A \wedge B) \to (B \wedge A)$.

Strengthening Implication

- So, as Figure 1 shows, introducing an instance of the connective $\to$ gives us a way to say in the logic that some premise leads to some conclusion.
- But notice that we'd ideally like to make a stronger claim than just $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$.
- That is, we want to be able to say not just that "starting from $\varphi \wedge \psi$, you can deduce $\psi \wedge \varphi$".
- We'd like to have our logic be capable of deriving the fact that $\varphi \wedge \psi$ and $\psi \wedge \varphi$ are *equivalent* statements.
- Remembering that we already have a way to state equivalence in our logic via the biimplicational connective $\leftrightarrow$, we add more logical rules.

Introducing ↔

Inference Rule 9 (Biimplication Introduction)**.**

$$\dfrac{\Gamma \vdash \varphi \to \psi \qquad \Delta \vdash \psi \to \varphi}{\Gamma, \Delta \vdash \varphi \leftrightarrow \psi}\ (\leftrightarrow\mathrm{I})$$

- With Rule 9, it's easy to see why the symbol ↔ was chosen to represent biimplication.
- It's because a biimplication essentially says "with either side (the antecedent) being true, you get the other side (the consequent) being true."
- The reason biimplication is used to capture equivalence, as our truth tables say, is that if one is true (false) then the other is also true (false).
- There are also elimination rules for ↔ that let us use equivalences in proofs.

Eliminating ↔

Inference Rule 10 (Biimplication Elimination 1)**.**

$$\dfrac{\Gamma \vdash \varphi \leftrightarrow \psi}{\Gamma \vdash \varphi \to \psi}\ (\leftrightarrow\mathrm{E}_1)$$

Inference Rule 11 (Biimplication Elimination 2)**.**

$$\dfrac{\Gamma \vdash \varphi \leftrightarrow \psi}{\Gamma \vdash \psi \to \varphi}\ (\leftrightarrow\mathrm{E}_2)$$

Proof of a Well-known Equivalence

- Now we can actually *prove* that $A \land B$ is equivalent to $B \land A$ (Figure 2 gives this proof).

$$\dfrac{\dfrac{\dfrac{\dfrac{A \land B \vdash A \land B}{A \land B \vdash B}\ (\land\mathrm{E}_2) \qquad \dfrac{A \land B \vdash A \land B}{A \land B \vdash A}\ (\land\mathrm{E}_1)}{A \land B \vdash B \land A}\ (\land\mathrm{I})}{\vdash (A \land B) \to (B \land A)}\ (\to\mathrm{I}) \qquad \dfrac{\dfrac{\dfrac{B \land A \vdash B \land A}{B \land A \vdash A}\ (\land\mathrm{E}_2) \qquad \dfrac{B \land A \vdash B \land A}{B \land A \vdash B}\ (\land\mathrm{E}_1)}{B \land A \vdash A \land B}\ (\land\mathrm{I})}{\vdash (B \land A) \to (A \land B)}\ (\to\mathrm{I})}{\vdash (A \land B) \leftrightarrow (B \land A)}\ (\leftrightarrow\mathrm{I})$$

Figure 2: Proof of $(A \land B) \leftrightarrow (B \land A)$.

Things to Note

- Notice that, in the proof given in Figure 2, there are no premises left of the turnstile.
- This means that what we've proved, namely that $A \land B$ and $B \land A$ are equivalent to one another, is not contingent on any other assumptions. This is exactly what we want our logic to say about equivalences.
- One technical note: I have used A and B in the proof in Figure 2. But a similiar proof would work for any two propositions, not just atomic ones.

Homework

Exercises

Problem 1. Starting with the assumptions $A \leftrightarrow B$, $(B \land A) \to C$, and A, give a sequent-style natural deduction proof of $A \to C$.