Introduction to Trees

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Review of Chains

- Recall that a **chain** is an order where any two distinct elements a and b are comparable (i.e. either $a \sqsubseteq b$ or $b \sqsubseteq a$).
- Recall also that in a chain, a is minimal (maximal) in a subset S iff it is least (greatest) in S.

A Finite Order has a Maximal/Minimal Element

Theorem 7.1: Any nonempty finite order has a minimal (and so, by duality, a maximal) member.

Proof. Let T be the set of natural numbers n such that every ordered set of cardinality $n+1$ has a minimal member, and show that T is inductive. □

A Nonempty Finite Chain has a Bottom/Top

Corollary 7.1: Any nonempty finite chain has a bottom(and so, by duality, a top).

Proof. This follows from the preceding theorem together with the fact just reviewed that in a chain, a member is least (greatest) iff it is minimal (maximal). □

A Finite Chain is Order-Isomorphic to a Natural

Theorem 7.2: For any natural number n, any chain of cardinality n is order-isomorphic to the usual order on n (i.e. the restriction to n of the usual $\leq$ order on ω).

Proof. By induction on n. The case $n = 0$ is trivial.

By inductive hypothesis, assume the statement of the theorem holds for the case $n = k$.

Let A of cardinality $k + 1$ be a chain with order $\sqsubseteq$.

By the Corollary, A has a greatest member a, so there is an order isomorphism f from k to $A \setminus \{a\}$.

The rest of the proof consists of showing that the function $f \cup \{< k, a >\}$ is an order isomorphism. $\square$

Finite Orders and their Covering Relations

Theorem 7.3: If $\sqsubseteq$ is an order on a finite set A, then $\sqsubseteq = \prec^*$.

Proof. That $\prec^* \subseteq \sqsubseteq$ follows easily from the transitivity of $\sqsubseteq$.

To prove the reverse inclusion, suppose $a \neq b$, $a \sqsubseteq b$ and let X be the set of all subsets of A which, when ordered by $\sqsubseteq$, are chains with b as greatest member and a as least member. Then X is nonempty since one of its members is $\{a, b\}$.

Then X itself is ordered by $\subseteq_X$, and so by Theorem 1 has a maximal member C.

Let $n+1$ be $|C|$; by Theorem 2, there is an order-isomorphism $f \colon n+1 \to C$. Clearly $n > 0$, $f(0) = a$, and f(n) = b.

Also, for each $m < n$, $f(m) \prec f(m+1)$, because otherwise, there would be a c properly between $f(m)$ and $f(m+1)$, contradicting the maximality of C. $\square$

Trees

A **tree** is a finite set A with an order $\sqsubseteq$ and a top $\top$, such that the covering relation $\prec$ is a function with domain $A \setminus \{\top\}$.

Tree Terminology

- The members of A are called the **nodes** of the tree.
- $\top$ is called the **root**.
- If $x \sqsubseteq y$, y is said to **dominate** x; and if additionally $x \neq y$, then y is said to **properly dominate** x.
- If $x \prec y$, then y is said to **immediately dominate** x; $y = \prec (x)$ is called the **mother** of x; and x is said to be a **daughter** of y.
- Distinct nodes with the same mother are called **sisters**.
- A minimal node (i.e. one with no daughters) is called a **terminal** node.
- A node which is the mother of a terminal node is called a **preterminal** node.

A Node Can't Dominate One of its Sisters

Theorem 7.4: In a tree, no node can dominate one of its sisters.

Proof. Exercise. $\square$

The $\uparrow$ Notation

If $\langle A, \sqsubseteq\rangle$ is a preordered set $a \in A$, we denote by $\uparrow a$ the set of upper bounds of $\{a\}$, i.e.

$$\uparrow a = \{x \in A \mid a \sqsubseteq x\}$$

In a Tree, $\uparrow a$ is Always a Chain

Theorem 7.5: For any node a in a tree, $\uparrow a$ is a chain.

Proof. Use RT to define a function $h\colon \omega \to A$, with $X = A$, $x = a$, and F the function which maps non-root nodes to their mothers and the root to itself.

Now let $Y = \mathsf{ran}(h)$; it is easy to see that Y is a chain, and that $Y \subseteq \uparrow a$.

To show that $\uparrow a \subseteq Y$, assume $b \in \uparrow a$; we'll show $b \in Y$.

By definition of $\uparrow a$, $a \sqsubseteq b$, and so by Theorem 3, $a \prec^* b$.

So there is $n \in \omega$ such that $a \prec_n b$, where $\prec_n$ is the n-fold composition of $\prec$ with itself.

I.e., there is an A-string $a_0 \dots a_n$ such that $a_0 = a$, $a_n = b$, and for each $k < n$, $a_k \prec a_{k+1}$.

But then $b = h(n)$, so $b \in Y$. □

When do Two Nodes in a Tree have a GLB?

Corollary 7.2: Two distinct nodes in a tree have a glb iff they are comparable.

Proof. Exercise. □

A Tree is an Upper Semilattice

Theorem 7.6: Any two nodes have a lub (and so a tree is an upper semilattice).

Proof. Exercise. □

Ordered Trees

- An **ordered tree** is a set A with *two* orders $\sqsubseteq$ and $\leq$, such that the following three conditions are satisfied:
  - A is a tree with respect to $\sqsubseteq$.
  - Two distinct nodes are $\leq$-comparable iff they are not $\sqsubseteq$ comparable.
  - (No-tangling condition) If a, b, c, d are nodes such that $a < b$, $c \prec a$, and $d \prec b$, then $c < d$.
- In an ordered tree, if $a < b$, then a is said to **linearly precede** b.

The Daughters of a Node Form a Chain

Theorem 7.7: If a is a node in an ordered tree, then the set of daughters of a ordered by $\leq$ is a chain.

Proof. Exercise. $\square$

The Terminal Nodes of an Ordered Tree Form a Chain

Theorem 7.8: In an ordered tree, the set of terminal nodes ordered by $\leq$ is a chain.

Proof. Exercise. $\square$

CFG Review

- Recall that a **CFG** is an ordered quadruple $\langle T, N, D, P \rangle$ where
  - T is a finite set called the **terminals**;
  - N is a finite set called **nonterminals**
  - D is a finite subset of $N \times T$ called the **lexical entries**;
  - P is a finite subset of $N \times N^+$ called the **phrase structure rules** (PSRs).
- Recall also these notational conventions:
  - '$A \to t$ ' means $\langle A, t \rangle \in D$.
  - '$A \to A_0 \ldots A_{n-1}$' means $\langle A, A_0 \ldots A_{n-1} \rangle \in P$.
  - '$A \to \{s_0, \ldots s_{n-1}\}$' abbreviates $A \to s_i$ $(i < n)$.

Phrase Structures for a CFG

- A **phrase structure** for a CFG $\mathbf{G} = \langle T, N, D, P \rangle$ is an ordered tree together with a **labelling** function $\mathbf{l}$ from the nodes to $T \cup N$ such that, for each node a,
  - $\mathbf{l}(a) \in T$ if a is a terminal node, and
  - $\mathbf{l}(a) \in N$ otherwise.
- Given a phrase structure with linearly ordered (as per Theorem 8) set of terminal nodes $a_0, \ldots, a_{n-1}$ with labels $t_0, \ldots, t_{n-1}$ respectively, the string $t_0 \ldots t_{n-1}$ is called the **terminal yield** of the phrase structure.

Weak and Strong Generative Capacity

- A phrase structure tree is **generated** by the CFG $\mathbf{G} = \langle T, N, D, P \rangle$ iff
  - for each preterminal node with label A and (terminal) daughter with label t, $A \to t \in D$; and
  - for each nonterminal nonpreterminal node with label A and linearly ordered (as per Theorem 7) daughters with labels $A_0, \ldots, A_{n-1}$ respectively, $(n > 0)$, $A \to A_0 \; \ldots A_{n-1} \in P$.
- The **strong generative capacity** of $\mathbf{G}$ is the set of phrase structures that it generates.
- The **weak generative capacity** of $\mathbf{G}$ is the function $\mathbf{wgc} : N \to T^*$ that maps each nonterminal symbol A to the set of T-strings which are terminal yields of phrase structures generated by $\mathbf{G}$ with root label A.