Introduction to Set Theory

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Introducing Sets

Let's suppose there are:

- things which we call **sets**, and
- a relationship between sets called **membership**.

Some Basic Terminology and Notation

- We use italic letters as names of arbitrary sets.
- We write '$A \in B$' to express that A is a member of B.
- We write '$A \notin B$' to express that A is not a member of B.
- If $A \in B$, we call A a **member**, or **element**, of B.
- Another way to say that $A \in B$ is to say A **belongs to** B.

Informal Set Theory

- We will make some basic assumptions about membership.
- We usually express our assumptions in ordinary English.
- But often, to avoid ambiguity, we use a special-purpose English-like language that we call **Mathese**.
- The assumptions we make about membership, together with the statements that follow from them by valid arguments, we call **informal set theory**.
- For now we won't try to make precise what we mean by 'valid arguments'.

Axiomatic Set Theory (1/2)

- Soon we will introduce a special symbolic language, **first-order logic (FOL)**, that will let us make statements about membership more precisely and more concisely.
- Mathese is a spoken approximation of FOL.
- We call the FOL counterparts of sentences **formulas**.
- We call the formulas that express assumptions **axioms**.

Axiomatic Set Theory (2/2)

- We will also see how to formalize the notion of 'valid argument' within FOL.
- Such formalized arguments are called **proofs**.
- And turning things around, valid arguments in English or Mathese are often called **informal proofs**.
- We call formulas that can be proved from the axioms **theorems**.
- Our axioms, together with the theorems we can prove from them, we will call **axiomatic set theory**.

Some Words of Caution

- Set theory will not tell you what sets and membership are; they are **unanalyzed primitives** of set theory.
- For now, you might find it helpful to think of a set as something like an invisible basket, and its members as something like marbles in the basket, but this analogy will only carry you so far.
- The assumptions we will make about membership are not the only possible assumptions one might make; our set theory is *a* set theory, not *the* set theory.

Before We Start

- We'll start with the least controversial assumptions.
- For those of you who already know FOL, we'll write below each assumption the corresponding FOL axiom.
- Don't worry if you don't know FOL; we'll fix that soon.

Assumption 1: Extensionality

English: If A and B have the same members, then they are the same set.

Mathese: For all x, for all y, if for all z, z is a member of x iff z is a member of y, then x equals y.

FOL: $\forall x \forall y((\forall z(z \in x \leftrightarrow z \in y)) \rightarrow x = y)$

Note 1: The intuition behind Extensionality is that, once you know what members a set has, you know which set it is.

Note 2: But there is nothing in our set theory so far that guaranteees that there actually *are* any sets.

Definitions: Subset and Proper Subset

- If every member of A is a member of B, we say that A is a **subset** of B, or **included in** B, written '$A \subseteq B$'. If not, we write '$A \nsubseteq B$.'

  Note 1: if $A \subseteq B$ and $B \subseteq A$, then it follows from Extensionality that $A = B$.

  Note 2: for any set A, $A \subseteq A$.

- If $A \subseteq B$ but $A \neq B$ then we say A is a **proper** subset of B, written '$A \subsetneq B$'.

Assumption 2: Empty Set

English: There is a set with no elements.

Mathese: There exists x such that, for all y, y is not a member of x.

FOL: $\exists x \forall y(y \notin x)$

Notation for the Empty Set

- By Extensionality, there can be only one set with no elements. We call it the **empty** set, written '$\emptyset$'.
- This is our first example of a commonplace practice in set theory: once we establish that there is exactly one set that has a given property (or equivalently, meets a certain description), then we can make up a name for it.
- Soon we will see that it is possible to do arithmetic within set theory, and that when we do so, it turns out that $\emptyset$ and the number 0 are the same thing. So we use '0' as a synonym for '$\emptyset'$.
- Obviously, for every set A, $\emptyset \subseteq A$.

Why are We Doing This?

- We aren't doing set theory just to kill time.
- We are doing it because we are going to *use* it to construct precise (although abstract) **models** of empirical linguistic phenomena (such as linguistic expressions, prosodic tunes, meanings, utterance contexts, etc.).
- To put it another way, set theory will be our workspace for linguistic modelling.

We Need More Sets

- In order for set theory to serve as our linguistic modelling workspace, we need for it to make plenty of sets available.
- But so far, the only set we 'have' is $\emptyset$.
- For example, we have no way to make a valid argument that there's a set with just one member, namely $\emptyset$.
- We will 'get' more sets the same we got $\emptyset$: by willing them into existence.
- And the way we do that is by making more assumptions.

Assumption 3: Pairing

English: If A and B are sets, then there is a set whose only members are A and B.

Mathese: For all x, for all y, there exists z such that x is a member of z, y is a member of z, and for all w, if w is a member of z, then either w equals x or w equals y.

FOL:

$\forall \mathrm{x} \forall \mathrm{y} \exists \mathrm{z}((x \in z) \wedge (y \in z) \wedge \forall w((w \in z) \rightarrow ((w = x) \vee (w = y))))$

Curly Bracket Notation (1/2)

- Because of Extensionality again, there is *only* one set whose only members are A and B, called the **(unordered) pair** of A and B, written '$\{A, B\}$',
- We could just as well have called this set $\{B, A\}$.
- Nothing rules out the possibility that A and B are the same set, so it follows from pairing that for any set A there is a set whose only member is A, namely $\{A, A\}$.
- We might as well just call that set $\{A\}$ rather than $\{A, A\}$.
- Such a set, with exactly one member, is called a **singleton**.

Curly Bracket Notation (2/2)

- More generally, we notate a nonempty finite set by listing its members, separated by commas, between curly brackets.
- We postpone getting clear about exactly what we mean by 'finite' and just rely on intuition for the time being.
- The order in which the members are listed doesn't matter.
- It doesn't make any sense to talk about *what order* the members of a set come in.
- Repetitions inside curly brackets don't matter either.
- It doesn't make any sense to talk about *how many times* one set is a member of another.

This Could be the Start of Something Big

- Remember 0 is a synonym for $\emptyset$.
- Now consider the singleton set $\{0\}$, which we call 1.
- Next, consider the set $\{0, 1\}$, which we call 2.
- Notice that 0 has zero members, 1 has one member, and 2 has two members.
- Notice also that 1 has 0 as a member, and that 2 has 0 and 1 as members.

What's 3?

- The obvious next step would be to say that 3 is $\{0, 1, 2\}$.
- But we have no way to make a valid argument that there actually *is* a set whose only members are 0, 1, and 2.
- Looks like it's time to make another assumption.

Assumption 4: Union

English: If A is a set, then there is a set whose members are those sets which are members of some member of A.

Mathese: For all x, there exists y such that, forall z, z is a member of y iff there exists w such that w is a member of x and z is a member of w.

FOL: $\forall x \exists y \forall z (z \in y \leftrightarrow (\exists w ((w \in x) \wedge (z \in w))))$

Notation for Union

- The set whose members are those sets which are members of some member of A is called the **union** of A, written '$\bigcup A$'.
- If $A = \{B, C\}$, then $\bigcup A$ is the set each of whose members is in either B or C (or both), written '$B \cup C$'.
- Note that in general $B \cup C$ is not the same set as $\{B, C\}$.

Three and Beyond

- For example, compare $2 \cup \{2\}$ with $\{2, \{2\}\}$.
- $\{2, \{2\}\}$ only has two members, namely 2 and $\{2\}$.
- Whereas $2 \cup \{2\}$ has three members:

$$2 \cup \{2\} = \{0, 1\} \cup \{2\} = \{0, 1, 2\}$$

- Hey! That's the set we wanted to call '3'!
- We can use this same trick over and over to keep getting more and more new sets.

Definition: Successor

- For any set A, the **successor** of A, written '$s(A)$', is the set $A \cup \{A\}$.
- That is, $s(A)$ is the set with the same members as A, except that A itself is also a member of $s(A)$.
- Nothing we have said rules out the possibility that $A \in A$, in which case $A = s(A)$.
- The most widely used set theory (called **Zermelo- Fraenkel** set theory, or just **ZF**) includes an assumption (called **Foundation**) which does rule out this possibility.
- But we will not assume Foundation in this book.

A Preview of the Natural Numbers

- Notice that 1 is the successor of 0, 2 is the successor of 1, and 3 is the successor of 2.
- Intuitively, the sets $0, 1, 2, 3, \ldots$ we get by starting with 0 and 'taking successors forever' are the natural numbers.
- But what *are* natural numbers? We'll come back to that.
- Do they form a set? We'll come back to that too.

Assumption 5: Powerset

English: If A is a set, then there is a set whose members are the subsets of A.

Mathese: For all x, there exists y such that, forall z, z is a member of y iff z is a subset of x.

FOL: $\forall x \exists y \forall z (z \in y \leftrightarrow (z \subseteq x))$

Notation for Powersets

- By Extensionality again, for any set A there can only be one sets whose members are the subsets of A.
- That set is called the **powerset** of A, written '$\wp(A)$'.
- Notice that $\wp(A)$ is not in general the same set as A, because usually the subsets of a set are not the same as the members of the set.
- For example, $0 \subseteq 0$, but $0 \notin 0$.

Definitions vs. Assumptions

- There's a crucial difference between the notions of successor and powerset.
- The successor of A is *defined* in terms of things whose existence can already be established on the basis of previous assumptions (singletons, unions); whereas the existence of the powerset of A is *assumed*.
- Why didn't we just *define* $\wp(A)$ to be the set whose members are the subsets of A?
- It's because nobody has found a valid argument (based on just the first four assumptions) that there *is* such a set!
- More generally, for an arbitrary condition on sets $P[x]$, there is no guarantee that there is a set whose members are all the sets x such that $P[x]$.
- The first person to realize this was the philosopher and mathematician Bertrand Russell, in 1902.

Russell's Paradox

Let $P[x]$ be the condition 'x is not a member of itself'. Russell showed that there cannot be a set whose members are all the sets x such that $P[x]$.

a. Suppose R were such a set.

b. Then either (i) R is a member of itself, or (ii) it isn't. Let's consider both possibilities.

c. *Possibility 1 ($R \in R$)*: then $R \notin R$, since the only members of R are sets which are *not* members of themselves.

d. *Possibility 1 ($R \notin R$)*: then R is not a member itself, so that it is a member of R.

e. Either way leads to a contradiction.

f. So the assumption must have been false that there is a set whose members are those sets which are not members of themselves.

A Bad Set-Theoretic Assumption Bites the Dust

- Russell's Paradox shows we don't have the option of adding the following to our set theory:

  Tentative Assumption: Comprehension

  For any condition $P[x]$ there is a set whose members are all the sets x such that $P[x]$.

- A more modest assumption is usually adopted instead.

Assumption 6: Separation
For any set A and any condition $P[x]$, there is a set whose members are all the x in A that satisfy $P[x]$.

- So far, assuming Separation has not been shown to lead to a contradiction.
- Separation is so-called because, intuitively, we are separating out from A some members that are special in some way, and collecting them together into a set.
- By Extensionality, there can be only one set whose members are all the sets x in A that satisfy $P[x]$.
- We call that set $\{x \in A \mid P[x]\}$.

Intersection

- In naive introductions to set theory, the **intersection** of two sets A and B, written '$A \cap B$', is often 'defined' as the set whose members are those sets which are members of both A and B.
- But how do we know there is such a set?
- If we assume Separation and take $P[x]$ to be the condition $x \in B$, then we can (unproblematically) define $A \cap B$ to be $\{x \in A \mid x \in B\}$.
- A and B are said to **intersect** provided $A \cap B$ is nonempty.
- Otherwise, A and B are said to be **disjoint**.
- A set is called **pairwise disjoint** if no two distinct members of it intersect.

Set Difference

- For two sets A and B, if we take $P[x]$ to be the condition $x \notin B$, then Separation guarantees the existence of the set $\{x \in A \mid x \notin B\}$.
- This set is called the **set difference** of A and B, or alternatively the **complement** of B **relative to** A, written '$A \setminus B$'.

There is No Universal Set

- A set is called **universal** if every set is a member of it.
- We can prove in our set theory that there is no universal set
- For suppose A were a universal set. Let $P[x]$ be the condition $x \notin x$. Then by Separation, there must be a set $\{x \in A \mid x \notin x\}$. But Russell's argument showed that there can be no such set. So the assumption that there was a universal set must have been false.

Definition: Ordered Pair

- If A and B are sets, we call the set $\{\{A\}, \{A, B\}\}$ the **ordered pair** of A and B, also written '$\langle A, B\rangle$'.
- $\langle A, B\rangle$ differs from $\{A, B\}$ in the crucial respect that no matter what A and B are, $\{A, B\} = \{B, A\}$, but $\langle A, B\rangle = \langle B, A\rangle$ only if $A = B$.
- More generally, if A, B, C, and D are sets, then $\langle A, B\rangle = \langle C, D\rangle$ only if $A = C$ and $B = D$.
- If C is the ordered pair of A and B, A is called the **first component** of C, and B is called the **second component** of C.

Definition: Cartesian Product

- For any sets A and B, there is a set whose members are all those sets which are ordered pairs whose first component is in A and whose second component is in B. (It's instructive to try to prove this. Hint: use Separation.)
- By Extensionality there can be only one such set. It is called the **cartesian product** of A and B, written '$A \times B$'.
- For any sets A, B, C, and D, $A \times B = C \times D$ only if $A = C$ and $B = D$. (Try to prove this.)
- A is called the **first factor** of $A \times B$, and B the **second factor**.

Definition: Ordered Triple

- The **ordered triple** of A, B, and C, written '$\langle A, B, C\rangle$', is defined to be the ordered pair whose first component is $\langle A, B\rangle$ and whose second component is C.
- Then A, B, and C are called, respectively, the **first**, **second**, and **third components** of $\langle A, B, C\rangle$.
- The **(threefold) cartesian product** of A, B, and C, written '$A \times B \times C$', is defined to be $(A \times B) \times C$. This is the set of all ordered triples whose first, second, and third components are in A, B, and C respectively.
- The definitions can be extended to ordered quadruples, quintuples, etc., and to n-fold cartesian products for $n > 3$, in an obvious way.

Definition: Cartesian Power

For any set A, a **cartesian power** of A is a cartesian product all of whose factors are A.

- The **first cartesian power** of A, written '$A^{(1)}$', is just A.
- The **cartesian square** of A, written '$A^{(2)}$', is $A \times A$.
- The **cartesian cube** of A, written '$A^{(3)}$', is $A \times A \times A$
- More generally, for $n > 3$, the **n-th cartesian power** of A, written '$A^{(n)}$', is the n-fold cartesian product all of whose factors are A.
- Additionally, the **zero-th cartesian power** of A, written '$A^{(0)}$', is defined to be the set 1 ($= \{\emptyset\}$).
- This last definition is closely related to the arithmetic fact that for any natural number n, $n^0 = 1$, but we postpone the explanation.