Relations

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Relations (Intuitive Idea)

- Intuitively, a *relation* is “the kind of thing that either holds or doesn’t hold between certain things.”
- *Examples:*
  - *Being less than* is a relation between two numbers.
  - *Loving* is a relation between two people.
  - *Owning* is a relation between a person and a thing.
  - *Being at* is a relation between a thing and a location.
  - *Knowing that* is a relation between a person and a proposition.

The Extension of a Relation (Intuitive Idea, 1/3)

- The *extension* of a relation is the set of ordered pairs $\langle x, y \rangle$ such that x is in the relation with y.
- For example, the extension of the love relation is the set of ordered pairs $\langle x, y \rangle$ such that x loves y.
- In general, which pairs are in the extension of a relation is *contingent*, i.e. depends on how things happen to be.
- For example, the way things actually are, Brad loves Angelina (let's say). But they could have been otherwise.

The Extension of a Relation (Intuitive Idea, 2/3)

- *Different* relations can have the *same* extension.
- *Example:* suppose it *just so happened* that for all pairs of people x and y, x loves y iff x's social security number is less than y's social security number.
- However, we wouldn't then say that loving someone is *the same thing* as having a lower social security number than that person.
- More generally, in natural language semantics, it's very important to distinguish between the *sense* of the word *love*, which is the love relation itself, and the *reference* of the word *love*, which is the extension of that relation.
- We postpone the question of how to model relations themselves (as opposed to their extensions) until we've introduced the semantic notion of a **proposition** (roughly: what a declarative sentence expresses).

The Extension of a Relation (Intuitive Idea, 3/3)

- *Mathematical* relations (such as being less than) differ from relations such as loving, owning, being at, or knowing that, in this important respect: *which ordered pairs are in the relation is not contingent.*
- For example,it doesn't *just so happen* that $2 < 3$; rather. things couldn't have been otherwise.
- Another way to say this is that 2 is *necessarily* less than 3 (not merely *contingently* less than 3).
- Since, with mathematical relations, which ordered pairs are in the relation is a matter of necessity (and not of contingency), mathematicians don't bother to make a distinction between a relation and its extension.
- So the idea of relation we are about to introduce will work fine for math, but when we start to discuss linguistic meaning, we will have to rethink things.

Preliminary Definition: Relation

- A **relation from** A **to** B, also called a **relation between** A **and** B, is a subset of $A \times B$.
- A **relation on** A is a relation between A and A, i.e. a subset of $A^{(2)}$.

Note: if R is a relation, we usually write $a\ R\ b$ as a shorthand for $\langle a, b \rangle \in R$.

Some Important Relations

- For any set A, the **identity** relation

$$\mathsf{id}_A =_{\text{def}} \{\langle x, y\rangle \in A \times A \mid x = y\}$$

  is a relation on A.
- For any set A, the **subset inclusion** relation

$$\subseteq_A =_{\text{def}} \{\langle x, y\rangle \in \wp(A) \times \wp(A) \mid x \subseteq y\}$$

  and the **proper subset inclusion** relation

$$\subsetneq_A =_{\text{def}} \{\langle x, y\rangle \in \wp(A) \times \wp(A) \mid x \subsetneq y\}$$

  are relations on $\wp(A)$.
- The **less than** relation

$$< =_{\text{def}} \{\langle m, n\rangle \in \omega \times \omega \mid m \subsetneq n\}$$

  is a relation on ω.

Definition: Inverse of a Relation

- If R is a relation from A to B, the **inverse** of R is the relation from B to A defined as follows:

$$R^{-1} =_{\text{def}} \{\langle x, y\rangle \in B \times A \mid y \; R \; x\}$$

- *Examples*:
  - $<^{-1} = >$
  - $\subseteq_A^{-1} = \supseteq_A$
  - $\mathsf{id}_A^{-1} = \mathsf{id}_A$
  - For any relation R, $(R^{-1})^{-1} = R$.

Definition: Composition of Relations

- Suppose R is a relation from A to B and S is a relation from B to C. Then the **composition** of S and R is the relation from A to C defined by

$$S \circ R =_{\text{def}} \{\langle x, z\rangle \in A \times C \mid \exists y \in B(x\ R\ y \wedge y\ S\ z)\}$$

- *Obvious fact:* If R is a relation from A to B, then

$$\mathsf{id}_B \circ R = R = R \circ \mathsf{id}_A$$

Definitions: Domain and Range of a Relation

Suppose R is a relation from A to B. Then:

- the **domain** of R is:

$$\mathsf{dom}(R) =_{\text{def}} \{x \in A \mid \exists y \in B(x \; R \; y)\}$$

- the **range** of R is:

$$\mathsf{ran}(R) =_{\text{def}} \{y \in B \mid \exists x \in A(x \; R \; y)\}$$

Definition: Relations of any Arity

- We defined a relation to be a subset of a cartesian product $A \times B$. More precisely. this is a **binary** relation.
- We define a **ternary relation** among the sets A, B, and C to be a subset of the threefold cartesian product $A \times B \times C$; thus a ternary relation is a set of ordered triples.
- For $n > 3$, n-fold cartesian products and n-ary relations are defined in the obvious way.
- For any $n \in \omega$, we define an n-ary **relation on** A to be a subset of $A^{(n)}$.
- So a **unary** relation on A is a subset of $A^{(1)} = A$.
- And a **nullary** relation on A is a subset of $A^{(0)} = 1$, i.e. either 0 or 1.

Definitions: Comparability and Connexity

Suppose R is a binary relation on A.

- Distinct $a, b \in A$ are called (R-)**comparable** if either $a\ R\ b$ or $b\ R\ a$; otherwise, they are called **incomparable**.
- R is called **connex** iff a and b are comparable for all distinct $a, b \in A$.
- *Exercise:* Are any of the relations we've already introduced connex?

Definitions: Reflexivity and Irreflexivity

Suppose R is a binary relation on A.

- R is called **reflexive** if $a\ R\ a$ for all $a \in A$ (i.e. $\mathrm{id}_A \subseteq R$).
- R is called **irreflexive** if $a \not R\ a$ for all $a \in A$ (i.e. $\mathrm{id}_A \cap R = \emptyset$).
- *Exercise:* Are any of the relations we've already introduced reflexive? Irreflexive?

Definitions: Reflexive Closure and Irreflexive Interior

Suppose R is a binary relation on A.

- The **reflexive closure** of R is the relation $R \cup \mathrm{id}_A$.
- The **irreflexive interior** of R is the relation $R \setminus \mathrm{id}_A$

More Exercises

- Prove: a relation is reflexive iff it is equal to its reflexive closure, and irreflexive iff it is equal to its irreflexive interior.
- Prove: the reflexive closure of R is the intersection of the set of reflexive relations on A which have R as a subset.
- Prove: The irreflexive interior of R is the union of the set of irreflexive relations which are subsets of R.
- What are the reflexive closure and the irreflexive interior of id_A? Of $\subseteq_A$? Of $<$?

Definition: Symmetry, Asymmetry, and Antisymmetry

Suppose R is a binary relation on A.

- R is called **symmetric** if $a \; R \; b$ implies $b \; R \; a$ for all $a, b \in A$ (i.e. $R = R^{-1}$).
- R is called **asymmetric** if $a \; R \; b$ implies $b \not R \; a$ for all $a, b \in A$ (i.e. $R \cap R^{-1} = \emptyset$).
- R is called **antisymmetric** if $a \; R \; b$ and $b \; R \; a$ imply $a = b$ for all $a, b \in A$ (i.e. $R \cap R^{-1} \subseteq \mathrm{id}_A$).

More Exercises

- Which relations that we've discussed so far are symmetric? Asymmetric? Antisymmetric?
- Prove that a relation is asymmetric iff it is both antisymmetric and irreflexive.

Definitions: Transitivity and Intransitivity

Suppose R is a binary relation on A.

- R is called **transitive** if $a\ R\ b$ and $b\ R\ c$ imply $a\ R\ c$ for all $a, b, c \in A$ (i.e. $R \circ R \subseteq R$).
- R is called **intransitive** if $a\ R\ b$ and $b\ R\ c$ imply $a\ \not R\ c$ for all $a, b, c \in A$ (i.e. $(R \circ R) \cap R = \emptyset$).

Note: these concepts have nothing to do with the syntactic notions of transitive and intransitive verbs!

Exercise: Which relations that we've discussed so far are transitive? Intransitive?

Definition: Equivalence Relation

Suppose R is a binary relation on A.

- R is called an **equivalence** relation iff it is reflexive, transtive, and symmetric.
- If R is an equivalence relation, then for each $a \in A$ the (R-)**equivalence class** of a is

$$[a]_R =_{\text{def}} \{b \in A \mid a\ R\ b\}$$

  Usually the subscript is dropped when it is clear from context which equivalence relation is in question.
- The members of an equivalence class are called its **representatives**.
- If R is an equivalence relation, the set of equivalence classes, written A/R, is called the **quotient** of A by R.

More Exercises

- Which relations that we've discussed so far are equivalence relations?
- What are their equivalence classes?
- Prove that if R is an equivalence relation on A, then A/R is a **partition** of A, i.e. it is (i) pairwise disjoint, and (2) its union is A.

(Pre-)Orders and Induced Equivalence

- A **preorder** on a set A is a binary relation $\sqsubseteq$ ('less than or equivalent to') on A which is reflexive and transitive.
- An antisymmetric preorder is called an **order**.
- The equivalence relation $\equiv$ **induced** by the preorder is defined by $a \equiv b$ iff $a \sqsubseteq b$ and $b \sqsubseteq a$.
- If $\sqsubseteq$ is an order, then $\equiv$ is just the identity relation on A, and correspondingly $\sqsubseteq$ is read as 'less than or equal to'.

Important Examples of (Pre-)Orders

- Two important orders in set theory:
  - For any set A, $\subseteq_A$ is an order on $\wp(A)$.
  - $\leq$ is an order on ω.
- The most important relation in linguistic semantics is the the **entailment** preorder on propositions.

  Before discussing entailment, we have to introduce the things that it relates: *propositions*.

An Intuitive Introduction to Propositions (1/3)

- Earlier we noted that linguistic semanticists in general (following Frege (1892)) distinguish between the *sense* of (an utterance of) a linguistic expression and its *reference*.
- An expression's sense is independent of how things are.

  (Remember our example: the sense of the verb *love* is the love relation, whatever *that* is.)
- Whereas the reference of an expression is the *extension* of its sense, which in general depends on how things are.

  (Remember our example: the reference of the verb *love* is the set of ordered pairs $\langle x, y \rangle$ such that x loves y.)
- The things that can be the senses of declarative sentences are usually called **propositions**.
- What's the extension of a proposition? We'll return to that.
- What *are* propositions?

An Intuitive Introduction to Propositions (2/3)

- Something similar to the notion of proposition used here was first suggested by the mathematician/philosopher Bernard Bolzano (*Wissenschaftlehre*, 1837)—his term was *Satz an sich* 'proposition in itself'.
- They are expressed by declarative sentences.
- They are the 'primary bearers of truth and falsity'. (A sentence is only secondarily, or derivatively, true or false, depending on what proposition it expresses.)
- They are the the 'objects of the attitudes', i.e. they are the things that are known, believed, doubted, etc.
- They are not linguistic.
- They are not mental.
- They are outside space, time, and causality.

An Intuitive Introduction to Propositions (3/3)

- What proposition a sentence expresses depends on the context in which it is uttered.
- For now we have to postpone consideration of what contexts are and how to model them.
- Sentences in different languages, or different sentences in the same language, can express the same proposition.
- Whether a proposition is true or false in general depends on *how things are* (or, in other words, *the way things are*).

An Intuitive Introduction to Worlds (1/2)

- A **(possible) world** is a way things might be.
- Here we mean not just a snapshot at a particular time, but a whole history, stretching as far back and as far forward as things go.
- One of the worlds, called the **actual** world, or just **actuality**, is the way things *really* are (again, stretching as far back and as far forward as things go).

An Intuitive Introduction to Worlds (2/2)

There have been two main schools of thought about what worlds are and how they relate to propositions:

- the view (apparently first advocated by Wittgenstein (1921) and C.I. Lewis (1923) that worlds are certain sets of propositions, called *maximal consistent* sets.
- the view expressed by Carnap (1947) and Kripke (1963) that propositions are sets of possible worlds.
  - In Carnap's version, worlds are *complete state descriptions*, which are sets of sentences in some logical language.
  - Whereas in Kripke's version, worlds are *theoretical primitives* and so not subject to further analysis.

The Wittgenstein/Lewis Take on Worlds and Propositions

- The Wittgenstein/Lewis view (worlds are maximal consistent sets of propositions) fits naturally with the semantics for modal logic (largely based on mathematics invented by Marshall Stone in the 1930s) developed by Tarski and his collaborators in the 1940s-early 1950s.
- This view has been advocated by numerous philosphers, such as Robert Adams, Alvin Plantinga, William Lycan, and Peter Forrest.
- But scarcely any linguistic semanticists seem to be familiar with this view.
- We'll try to correct that imbalance.

The Carnap/Kripke Take on Worlds and Propositions

- Carnap's (1947) idea that propositions are sets of worlds is still the mainstream view among linguistic semanticists.
- However, his idea that worlds themselves are sets of sentences in a logical language ('complete state descriptions') was discarded in favor of Kripke's (1963) treatment of worlds as theoretical primitives.
- Kripke's view was subsequently advocated by certain philosophers—David Lewis, Robert Stalnaker, Richard Montague, and David Kaplan—who exerted a powerful influence on linguistic semanticists, such as Barbara Partee and David Dowty.
- But among philosophers, nowadays it seems that Stalnaker is the only one still defending this view.
- Soon we'll see why.

- We assume there is a set P of things we call **propositions**.
- We assume there is a set W of things we call **worlds**.
- We assume that there is a distinguished world $\mathbf{w} \in W$ called the **actual** world.
- We assume there is a relation @, called **holding**, between propositions and worlds.
- If $p@w$, we say p **holds at** w, or is **true at** w, or is a **fact** of w; otherwise, we say p is **false** at w.
- The theory unfolds differently depending on whether we develop it in accordance with the Wittgenstein/Lewis view or the Kripke view.
- We will consider both.

Kinds of Propositions

A proposition p is called:

- a **necessary truth**, or a **necessity**, iff $p@w$ for every world w
- a **truth** iff $p@\mathbf{w}$
- a **falsehood** iff $\neg(p@\mathbf{w})$
- a **necessary falsehood**, or an **impossibility**, iff $p@w$ for no world w
- a **possibility** iff $p@w$ for some world w
- **contingent** iff it is neither a necessary truth nor a necessary falsehood.

Intuitive Introduction to Entailment

- Most semanticists assume that there is a binary relation (in the mathematical sense) between propositions, called **entailment**.
- The basic intuition about entailment is that for two propositions p and q, p entails q just in case, no matter how things are, if p is true with things that way, then so is q.
- If sentence S_1 expresses p and sentence S_2 expresses q, then we also say S_1 **entails** S_2, or that S_2 **follows from** S_1, if p entails q.
- p and q (or S_1 and S_2) are called **(truth-conditionally) equivalent** iff they entail each other.

Formalizing Entailment

- We define the (binary) entails relation on propositions as follows: for all $p, q \in \mathrm{P}$, p entails q iff for every $w \in \mathrm{W}$, if $p@w$ then $q@w$.
- It's easy to see that entails is a preorder.
- We say p and q are **(truth-conditionally) equivalent** iff $p \equiv q$, where $\equiv$ is the equivalence relation induced by entailment.
- So p and q are equivalent iff they are true at the same worlds.

Formalizing Entailment *à la* Kripke (1/2)

- To formalize the Kripke view of worlds and propositions, we first assume that *propositions are the same thing as sets of worlds*, i.e.

$$\mathrm{P} =_{\mathrm{def}} \wp(W)$$

- Next, we define the holding relation between propositions and worlds as follows:

$$@ =_{\mathrm{def}} \{\langle p, w\rangle \in \mathrm{P} \times \mathrm{W} \mid w \in p\}$$

- From this it follows from the definition of entailment (previous slide) that entailment is just the inclusion relation on sets of worlds:

$$\mathsf{entails} = \subseteq_{\mathrm{W}}$$

Formalizing Entailment *à la* Kripke (2/2)

- An outstanding virtue of the Kripke view is how breathtakingly easy it is to model mathematically.
- Could the overwhelming popularity of this approach among linguistic semanticists have anything to do with this?
- Unfortunately, on the Kripke view, entailment is not just a preorder, but a *order*, i.e. it is not just reflexive and transitive but also antisymmetric. So if two propositions are equivalent, they are *the same proposition*.
- And so, if two *sentences* entail each other, they must have the same sense, a consequence that philosophers (Stalnaker excluded) generally find unacceptable.
- Linguists are aware of the problem, but for the most part stick with the Kripke view anyway. (Be prepared for this if you are planning to take Semantics next quarter.)

Formalizing Entailment *à la* Wittgenstein/Lewis

- To formalize the Wittgenstein/Lewis view of worlds and propositions, we first assume that worlds are certain sets of propositions, i.e.:

$$\mathrm{W} \subsetneq \wp(\mathrm{P})$$

- More specifically, we take worlds to be *maximal consistent* sets of propositions. Intuitively speaking, this means that:
  - a world has enough propositions to 'settle all questions', and
  - a world doesn't have any impossiblities (necessarily false propositions)
- But before we can say *exactly* what we mean by a maximal consistent set, we need to put a little more mathematical machinery in place.

More Definitions for Preorders

- Background assumptions:
  - $\sqsubseteq$ is a preorder on A
  - $\equiv$ is the induced equivalence relation
  - $S \subseteq A$
  - $a \in A$ (not necessarily $\in S$)
- We call a an **upper** (**lower**) **bound** of S iff, for every $b \in S$, $b \sqsubseteq a$ ($a \sqsubseteq b$).
- Suppose moreover that $a \in S$. Then a is said to be:
  - **greatest** (**least**) in S iff it is an upper (lower) bound of S
  - a **top** (**bottom**) iff it is greatest (least) in A
  - **maximal** (**minimal**) in S iff, for every $b \in S$, if $a \sqsubseteq b$ ($b \sqsubseteq a$), then $a \equiv b$.

Note: the definition of greatest/least above is equivalent to the one in Chapter 3.

Some Observations

- Background assumptions:
  - $\sqsubseteq$ is a preorder on A
  - $\equiv$ is the induced equivalence relation
  - $S \subseteq A$
- If S has any greatest (least) elements, then they are the only maximal (minimal) elements of S.
- All greatest (least) members of S are equivalent.
- And so all tops (bottoms) of A are equivalent.
- And so if $\sqsubseteq$ is an order, S has at most one greatest (least) member, and A has at most one top (bottom).
- Maximal (minimal) elements needn't be greatest (least).

(Pre-)Chains

- A connex (pre-)order is called a **(pre-)chain**.
- Chains are also called **total orders**, or **linear orders**.
- In a (pre-)chain, being maximal (minimal) in S is the same thing as being greatest (least) in S.
- A chain is called **well-ordered** iff every nonempty subset has a least element.
- It is possible to prove based on the set-theoretic assumptions we have alreay made that ω is well-ordered by the usual ($\leq$) order.

LUBs and GLBs

- Background assumptions:
  - $\sqsubseteq$ is a preorder on A
  - $S \subseteq A$
- Let UB(S) (LB(S)) be the set of upper (lower) bounds of S.
  - A least member of UB(S) is called a **least upper bound (lub)** of S.
  - A greatest member of LB(S) is called a **greatest lower bound (glb)** of S.

More about LUBs and GLBs

- Background assumptions:
  - $\sqsubseteq$ is a preorder on A
  - $S \subseteq A$
- Any greatest (least) member of S is a lub (glb) of S.
- All lubs (glbs) of S are equivalent.
- If $\sqsubseteq$ is an order, then S has at most one lub (glb).
- A lub (glb) of A is the same thing as a top (bottom).
- A lub (glb) of $\emptyset$ is the same thing as a bottom (top).