(Pre-)Lattices

Carl Pollard

Department of Linguistics
Ohio State University

November 17, 2011

Prelattices

- A **prelattice** is a preordered algebra $\langle P, \sqsubseteq, \sqcap, \sqcup \rangle$ where
  - $\langle P, \sqsubseteq, \sqcap \rangle$ is a lower semilattice and
  - $\langle P, \sqsubseteq, \sqcup \rangle$ is an upper semilattice.
- A **bounded prelattice** is a preordered algebra $\langle P, \sqsubseteq, \sqcap, \sqcup, \top, \bot \rangle$ where
  - $\langle P, \sqsubseteq, \sqcap, \sqcup \rangle$ is a prelattice
  - $\top$ is a top
  - $\bot$ is a bottom

Basic Facts about Prelattices (Review)

- $\sqcap$ and $\sqcup$ are:
  - monotonic in both arguments
  - associative u.t.e.
  - commutative u.t.e.
  - idempotent u.t.e.
- $\sqcap$ ($\sqcup$) is a glb (lub) operation
- Interdefinability: for all $p, q \in P$,

$$p \sqcap q \equiv p \text{ iff } p \sqsubseteq q \text{ iff } p \sqcup q \equiv q$$

- Absorption u.t.e.:

$$(p \sqcup q) \sqcap q \equiv q \equiv (p \sqcap q) \sqcup q;$$

More Facts about Prelattices

- **Semidistributivity**: For all $a, b \in P$:

$$(p \sqcap q) \sqcup (p \sqcap r) \sqsubseteq p \sqcap (q \sqcup r)$$

- A prelattice is called **distributive u.t.e** if the inequality reverse to Semidistributivity holds:

$$p \sqcap (q \sqcup r) \sqsubseteq (p \sqcap q) \sqcup (p \sqcap r)$$

  so that in fact

$$p \sqcap (q \sqcup r) \equiv (p \sqcap q) \sqcup (p \sqcap r)$$

- **Theorem**: a prelattice is distributive u.t.e. iff the following equivalence holds for all $a, b, c \in P$ (obtained from the one above by interchanging $\sqcap$ and $\sqcup$):

$$p \sqcup (q \sqcap r) \equiv (p \sqcup q) \sqcap (\mathrm{p} \sqcup r)$$

RPC Operations (Review)

- Let $\langle P, \sqsubseteq, \sqcap \rangle$ be a lower semilattice, and $\dashv$ a binary operation on A, such that for all $p, q, r \in P$:

  $$p \sqcap r \sqsubseteq q \text{ iff } r \sqsubseteq p \dashv q$$

  i.e. $p \dashv q$ is a greatest member of $\{r \in A \mid p \sqcap r \sqsubseteq q\}$ Then $\dashv$ is called a **relative pseudocomplement (rpc)** operation with respect to $\sqcap$.
- It can be shown that an rpc operation is antitonic on its first argument and monotonic on its second argument.

(Pseudo-)Complement Operations (Review)

- Suppose $\langle, P, \sqsubseteq, \sqcap, \bot \dashv\rangle$ is a lower presemilattice with a bottom $\bot$, , and $'$ is a unary operation on A such that, for all $p \in P$:

  $$p' \equiv p \dashv \bot$$

  Then $'$ is called a **pseudocomplement** operation, and p' is called the **pseudocomplement** of p.
- It is easy to show that $\bot'$ is a top.
- It is easy to show that, for all $p \in A$, $p \sqsubseteq (p')'$.
- If additionally, for all $p \in A$, $(p')' \sqsubseteq p$, so that

  $$(p')' \equiv p,$$

  then $'$ is called a **complement** operation, and p' is called the **complement** of p.

Preheyting Algebras

- A **pre-heyting algebra** is a preordered algebra $\langle P, \sqsubseteq, \sqcap, \sqcup, \dashv, \prime, \top, \bot \rangle$ where:
  - $\langle P, \sqsubseteq, \sqcap, \sqcup, \top, \bot \rangle$ is a bounded prelattice
  - $\dashv$ is an rpc operation
  - $\prime$ is a pseudocomplement operation.
- Preheyting algebras can be shown to be distributive u.t.e.
- The set of sentences of propositional logical forms a preheyting algebra with the preorder defined by

$$A \sqsubseteq B \text{ iff } A \vdash B \text{ where } \vdash \text{ is IPL provability.}$$

Preboolean Algebras

- A **pre-boolean algebra** is a pre-heyting algebra satisfying either of the following (equivalent!) conditions:
  - The pseudocomplement operation $'$ is a complement operation, i.e. for all $p \in P$,
    $$(p')' \equiv p,$$
  - For all $p \in P$, $p \sqcup p' \equiv \top$.
- The set of sentences of propositional logical forms a preboolean algebra with the preorder defined by
  $$A \sqsubseteq B \text{ iff } A \vdash B \text{ where } \vdash \text{ is CPL provability.}$$
- Under standard assumptions about how natural language entailment works, propositions (the kinds of things that can be senses expressed by declarative sentence utterances) form a preboolean algebra with entailment as the preorder.

The Preboolean Algebra of Propositions

- $\sqsubseteq$ (renamed entails) represents entailment
- $\sqcap$ (renamed and) represents the meaning of *and*
- $\sqcup$ (renamed or) represents the meaning of *or*
- $\dashv$ (renamed implies) represents the meaning of *if ... then*
- $'$ (renamed not) represents the meaning of *it is not the case that* or *no way*
- $\top$ (renamed truth) represents some necessarily true proposition
- $\bot$ (renamed falsity) represents some necessarily false proposition
- In the Lewis/Wittgenstein-style modelling, worlds are *maximal consistent sets* of propositions (yet to be defined).

Heyting Algebras and Boolean Algebras

- Predictably, a **heyting (boolean) algebra** is an antisymmetric preheyting (preboolean) algebra.
- An example of a heyting algebra is the set of open sets of real numbers ordered by inclusion. (Exercise: what are the operations?)
- The most familiar boolean algebras are power sets ordered by subset inclusion. (Exercise: what are the operations?)
- Special case: $2 = \wp(1) = \{0, 1\}$. Semanticists often call this the algebra of **truth values**, and rename 1 and 0 to **t** and **f** respectively.
- Special case: Under the Kripke/Montague-style modelling of propositions as sets of worlds, propositions form a boolean algebra with the entailment order being subset inclusion.

Worlds and Propositions *à la* Kripke/Montague

- We take worlds to be a set W of unanalyzed primitives
- We model propositions as sets of worlds: $\mathrm{P} =_{\mathrm{def}} \wp(\mathrm{W})$
- $p@w$ means $w \in p$, so entails is $\subseteq_W$
- and is intersection
- or is union
- implies is relative complement
- not is complement
- There is only one necessary truth.
- There is only one necessary falsehood.
- Sentences with the same truth conditions have the same meaning.

Toward Modelling Worlds as Maximal Consistent Sets

- We take worlds to be certain subsets of of the preboolean algebra of propositions, i.e. $\mathrm{W} \subsetneq \wp(\mathrm{P})$. (Which subsets? We'll come back to that.)
- $p@w$ means $p \in w$.
- For the preorder $\mathsf{entails}$ in P to be a good representation of entailment, it will have to be the case that for any two propositions p and q, p $\mathsf{entails}$ q iff for every world w, if $p \in w$ then $q \in w$.
- Turning things around, for any p and q such that it is *not* the case that p $\mathsf{entails}$ q, there must exist a w such that $p \in w$ but $q \notin w$.
- Informally: whatever worlds are, there have to be 'enough' of them.
- So which subsets of P should be in W?
- To answer this, we need to know more about certain special subsets of preboolean algebras.