(Pre-)Algebras

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Definition: Equivalence Relation

Suppose R is a binary relation on A.

- R is called an **equivalence** relation iff it is reflexive, transtive, and symmetric.
- If R is an equivalence relation, then for each $a \in A$ the (R-)**equivalence class** of a is

  $$[a]_R =_{\text{def}} \{b \in A \mid a\ R\ b\}$$

  Usually the subscript is dropped when it is clear from context which equivalence relation is in question.
- The members of an equivalence class are called its **representatives**.
- If R is an equivalence relation, the set of equivalence classes, written A/R, is called the **quotient** of A by R.

(Pre-)Orders and Induced Equivalence

- A **preorder** on a set A is a binary relation $\sqsubseteq$ ('less than or equivalent to') on A which is reflexive and transitive.
- An antisymmetric preorder is called an **order**.
- The equivalence relation $\equiv$ **induced** by the preorder is defined by $a \equiv b$ iff $a \sqsubseteq b$ and $b \sqsubseteq a$.
- If $\sqsubseteq$ is an order, then $\equiv$ is just the identity relation on A, and correspondingly $\sqsubseteq$ is read as 'less than or equal to'.

Important Examples of (Pre-)Orders

- Two important orders in set theory:
  - For any set A, $\subseteq_A$ is an order on $\wp(A)$.
  - $\leq$ is an order on ω.
- The most important relation in linguistic semantics is the the **entailment** preorder on propositions.

More Definitions for Preorders

- Background assumptions:
  - $\sqsubseteq$ is a preorder on A
  - $\equiv$ is the induced equivalence relation
  - $S \subseteq A$
  - $a \in A$ (not necessarily $\in S$)
- We call a an **upper** (**lower**) **bound** of S iff, for every $b \in S$, $b \sqsubseteq a$ ($a \sqsubseteq b$).
- Suppose moreover that $a \in S$. Then a is said to be:
  - **greatest** (**least**) in S iff it is an upper (lower) bound of S
  - a **top** (**bottom**) iff it is greatest (least) in A
  - **maximal** (**minimal**) in S iff, for every $b \in S$, if $a \sqsubseteq b$ ($b \sqsubseteq a$), then $a \equiv b$.

Some Observations

- Background assumptions:
  - $\sqsubseteq$ is a preorder on A
  - $\equiv$ is the induced equivalence relation
  - $S \subseteq A$
- If S has any greatest (least) elements, then they are the only maximal (minimal) elements of S.
- All greatest (least) members of S are equivalent.
- And so all tops (bottoms) of A are equivalent.
- And so if $\sqsubseteq$ is an order, S has at most one greatest (least) member, and A has at most one top (bottom).
- Maximal (minimal) elements needn't be greatest (least).

(Pre-)Chains

- A connex (pre-)order is called a **(pre-)chain**.
- Chains are also called **total orders**, or **linear orders**.
- In a (pre-)chain, being maximal (minimal) in S is the same thing as being greatest (least) in S.

LUBs and GLBs, MUBs and MLBs

- Background assumptions:
  - $\sqsubseteq$ is a preorder on A
  - $S \subseteq A$
  - $\mathrm{UB}(S)$ ($\mathrm{LB}(S)$) is the set of upper (lower) bounds of S.
- A least (minimal) member of $\mathrm{UB}(S)$ is called a **least (minimal) upper bound** or **lub (mub)** of S.
- A greatest (maximal) member of $\mathrm{LB}(S)$ is called a **greatest (maximal) lower bound** or **glb) (mlb)** of S.

More about LUBs and GLBs

- Background assumptions:
  - $\sqsubseteq$ is a preorder on A
  - $S \subseteq A$
- Any greatest (least) member of S is a lub (glb) of S.
- All lubs (glbs) of S are equivalent.
- If $\sqsubseteq$ is an order, then S has at most one lub (glb).
- A lub (glb) of A is the same thing as a top (bottom).
- A lub (glb) of $\emptyset$ is the same thing as a bottom (top).

Monotonicity, Antitonicity, and Tonicity

Suppose A and B are preordered by $\sqsubseteq$ and $\leq$ respectively. Then a function $f\colon A \to B$ is called:

- **monotonic** or **order-preserving** iff, for all $a, a' \in A$, if $a \sqsubseteq a'$, then $f(a) \leq f(a')$;
- **antitonic** or **order-reversing** iff, for all $a, a' \in A$, if $a \sqsubseteq a'$, then $f(a') \leq f(a)$; and
- **tonic** iff it is either monotonic or antitonic.

Preorder (Anti-)Isomorphism

- A monotonic (antitonic) bijection is called a **preorder isomorphism (preorder anti-isomorphism)** provided its inverse is also monotonic (antitonic).
- Two preordered sets are said to be **preorder-isomorphic** provided there is a preorder isomorphism from one to the other.

Algebras

- An **algebra** is a set A with one or more operations (where 'special elements' are thought of as nullary operations).
- Some of the simplest algebras are ones with just a single binary operation $\circ$. Some important examples:
  - **Semigroups**: $\circ$ is associative.
  - **Commutative semigroups**: $\circ$ is associative and commutative.
  - **Semilattices**: $\circ$ is associative, commutative, and **idempotent** (i.e. $a \circ a = a$ for all $a \in A$).
- A **monoid** is a semigroup with a two-sided identity element $\mathbf{e}$ (i.e. $a \circ \mathbf{e} = a = \mathbf{e} \circ a$ for all $a \in A$).

Examples of Monoids

- ω with $+$ as the operation and 0 as the identity for $+$.
- ω with $\cdot$ as the operation and 1 as the identity for $\cdot$
- For any set A, A^* with $\frown$ (**concatenation**) as the operation and ϵ_A (the null A-string) as the identity for $\frown$.

  Here if $f \in A^m$ and $g \in A^n$, $f \frown g \in A^{m+n}$ is given by

  - $(f \frown g)(i) = f(i)$ for all $i < m$; and
  - $(f \frown g)(m + i) = g(i)$ for all $i < n$.

Tonicity Generalized

- Recall: a unary operation on a (pre)order is called **tonic** provided it is either monotonic or antitonic.
- An operation of arbitrary arity on a (pre)order is called **tonic** if it is 'tonic in each argument as the other arguments are held fixed'.
  - All nullary operations are (trivially) tonic.
  - The two definitions coincide in the unary case.
  - a binary operation $\circ$ is tonic iff (1) for each a, the function that maps each b to $a \circ b$ is tonic, and (2) for each b, the function that maps each a to $a \circ b$ is tonic.

(Pre)ordered Algebras

- A **(pre)ordered algebra** is a (pre)order A which is also an algebra whose operations are all tonic.
- An operation in a preordered algebra is said to have a property **up to equivalence (u.t.e.)** if it holds with $=$ replaced by $\equiv$, where $\equiv$ is the equivalence relation induced by the preorder.
- For example, $\circ$ is commutative u.t.e. iff for all $a, b \in A$, $a \circ b \equiv b \circ a$.

Substitutivity u.t.e

- Preordered algebras enjoy the property of **substitutivity u.t.e**, i.e. replacing the arguments of any operation by equivalents yields an equivalent result.
- For example, in the binary case, this means that if $a \equiv b$ and $c \equiv d$, then $a \circ c \equiv b \circ d$.

Some Kinds of Preordered Algebras

For future reference:

- A **presemigroup** is a preorder with one binary operation $\circ$ which is monotonic on both arguments and associative u.t.e.
- A **presemilattice** is a presemigroup which is both commutative u.t.e. and idempotent u.t.e.
- A **premonoid** is a presemigroup with an additional unary operation **e** which is a two-sided identity u.t.e.

Ordered Algebras

A 'prewidget' is a called an 'ordered widget' iff it is antisymmetric. Examples:

- An **ordered semigroup** is an antisymmetric presemigroup.
- An **ordered semilattice** is an antisymmetric presemilattice.
- An **ordered monoid** is an antisymmetric premonoid.

An Important Example of an Ordered Monoid

For any set A, $\wp(A^*)$ forms a monoid with

- A-**languages** (i.e. sets of A-strings) as the elements
- $\bullet$ (**language concatenation**) as the binary operation, where for any A-languages L and M, $L \bullet M$, is the set of all strings of the form $u \frown v$ where $u \in L$ and $v \in M$
- $1_A = \{\epsilon_A\}$ as the identity for $\bullet$.

We turn this into an ordered monoid by taking the order to be subset inclusion of languages. (You need to check that $\bullet$ is monotonic in both arguments.)

Two Important Examples of an Ordered Semilattice

In both examples, we take the order to be the subset inclusion ordering on $\wp(A)$, for some set A.

- Example 1: take the binary operation to be set intersection.

  Observation: $a \subseteq b$ iff $a \cap b = a$.
- Example 2: take the binary operation to be set union.

  Observation: $a \subseteq b$ iff $a \cup b = b$.

These observations motivate the following definitions.

Two Kinds of Presemilattices

Suppose $\langle A, \sqsubseteq, \circ \rangle$ is a presemilattice, i.e. $\circ$ is monotonic in both arguments, associative u.t.e., commutative u.t.e., and idempotent u.t.e. Then it is called:

- **upper** iff, for all $a, b \in A$, $a \sqsubseteq b$ iff $a \circ b \equiv b$.
- **lower** iff, for all $a, b \in A$, $a \sqsubseteq b$ iff $a \circ b \equiv a$.

A Theorem about Presemilattices

- In an upper presemilattice, $\circ$ is a join (lub operation), hence usually written $\sqcup$.
- In a lower presemilattice, $\circ$ is a meet (glb operation), hence usually written $\sqcap$.

A Theorem about lubs and glbs

Suppose $\langle A, \sqsubseteq, \circ \rangle$ is a preorder with a join (meet) $\circ$.

Then it is an upper (lower) presemilattice, i.e. $\circ$ is tonic in both arguments, associative u.t.e., commutative u.t.e., idempotent u.t.e., and for all $a, b \in A$, $a \sqsubseteq b$ iff $a \circ b \equiv b$ ($a \circ b \equiv a$).

Relative Pseudocomplement (RPC) Operations

- Let $\langle A, \sqsubseteq, \sqcap \rangle$ be a lower semilattice, and $\dashv$ a binary operation on A, such that for all $a, b, c \in A$:

  $$a \sqcap c \sqsubseteq b \text{ iff } c \sqsubseteq a \dashv b$$

  i.e. $a \dashv b$ is a greatest member of $\{c \in A \mid a \sqcap c \sqsubseteq b\}$

  Then $\dashv$ is called a **relative pseudocomplement (rpc)** operation with respect to $\sqcap$.

- It can be shown that an rpc operation is antitonic on its first argument and monotonic on its second argument.

(Pseudo)complement

- Suppose $\langle, A, \sqsubseteq, \sqcap, \bot \dashv \rangle$ is a lower presemilattice with a bottom element $\bot$, and $'$ is a unary operation on A such that, for all $a \in A$:

  $$a' \equiv a \dashv \bot$$

  Then $'$ is called a **pseudocomplement** operation, and a' is called the **pseudocomplement** of a.

- If additionally, for all $a \in A$,

  $$(a')' \equiv a,$$

  then $'$ is called a **complement** operation, and a' is called the **complement** of a.