Positive Intuitionistic Propositional Logic

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Positive Intuitionistic Propositional Logic (PIPL)

- PIPL is like LL but with more connectives and rules.
- The only connectives of PIPL are
  - The 0-ary connective T (read 'true'), and
  - the two binary connectives $\rightarrow$ (intuitionistic implication) and $\wedge$ (conjunction).
  - PIPL underlies the *type systems* of typed lambda calculus (TLC) and higher order logic (HOL), which are used for notating both pheno and semantics in linear grammar.

Axioms of (Pure) PIPL

- Like LL, PIPL has the Hypothesis schema

$$A \vdash A$$

- In addition, it has the True axiom

$$\vdash \mathrm{T}$$

  Intuitively, T is usually thought of corresponding to an arbitrary necessary truth.

Rules of PIPL

- Introduction and elimination rules for implication
- Introduction and elimination rules for conjunction
- Two **structural** rules, Weakening and Contraction, which affect only the contexts of sequents

PIPL Rules for Implication

These are the same as for LL, but with $\multimap$ replaced by $\rightarrow$:

Modus Ponens, also called $\rightarrow$-Elimination:

$$\frac{\Gamma \vdash A \rightarrow B \qquad \Delta \vdash A}{\Gamma, \Delta \vdash B} \rightarrow\text{E}$$

Hypothetical Proof, also called $\rightarrow$-Introduction:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow\text{I}$$

PIPL Rules for Conjunction

The rules for conjunction include *two* elimination rules (for eliminating the left and right conjunct respectively):

$\wedge$-Elimination 1:

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge\text{E1}$$

$\wedge$-Elimination 2:

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge\text{E2}$$

$\wedge$-Introduction:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \wedge\text{I}$$

PIPL Structural Rules

Weakening:

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A} \text{W}$$

Intuitively: if we can prove something from certain assumptions, we can also prove it with more assumptions.

Contraction:

$$\frac{\Gamma, B, B \vdash A}{\Gamma, B \vdash A} \text{C}$$

Intuitively: repeated assumptions can be eliminated.

These may seem too obvious to be worth stating, but in fact they *must* be stated, because in some logics (such as LL) they are not available!

Extensions of PIPL

- By adding still more connectives—$\vee$ (disjunction), F (false), and $\neg$ (negation)—and corresponding rules/axioms to PIPL we get full intutionistic propositional logic (IPL).
- With the addition of one more rule we get classical propositional logic (CPL).
- And with the addition of rules for (universal and existential) quantification, we get (classical) first-order logic (FOL).

IPL Rules for Disjunction

The rules for disjunction include *two* introduction rules (for introducing the left and right conjunct respectively):

$\vee$-Elimination:

$$\frac{\Gamma \vdash A \vee B \qquad A, \Delta \vdash C \qquad B, \Theta \vdash C}{\Gamma, \Delta, \Theta \vdash C} \vee\text{E1}$$

$\vee$-Introduction 1:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee\text{I1}$$

$\vee$-Introduction 2:

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee\text{I2}$$

The IPL Axiom for False (F)

- The False Axiom

$$\text{F} \vdash A$$

  is traditionally called EFQ (*ex falso quodlibet*).
- Intuitively, F is usually thought of corresponding to an arbitrary impossibility (necessary falsehood).
- EFQ is easily shown to be equivalent to the following rule:

  F-Elimination:

$$\frac{\Gamma \vdash \text{F}}{\Gamma \vdash A} \text{FE}$$

IPL Rules for Negation

If we think of $\neg A$ as shorthand for $A \rightarrow \mathrm{F}$, then these rules are just special cases of Modus Ponens and Hypothetical Proof:

¬-Elimination:

$$\frac{\Gamma \vdash \neg A \qquad \Delta \vdash A}{\Gamma, \Delta \vdash \mathrm{F}} \neg\mathrm{E}$$

¬-Introduction, or Proof by Contradiction

$$\frac{\Gamma, A \vdash \mathrm{F}}{\Gamma \vdash \neg A} \neg\mathrm{I}$$

Another name for ¬E is *Indirect Proof.*

There are reasons (related to natural language semantics) to regard negation as a connective in its own right rather than as an abbreviatory convention.

Classical Propositional Logic (CPL)

CPL is obtained from IPL by the addition of any one of the following, which can be shown to be equivalent:

Reductio ad Absurdum:

$$\frac{\Gamma, \neg A \vdash \mathrm{F}}{\Gamma \vdash A} \mathrm{RAA}$$

Double Negation Elimination:

$$\frac{\Gamma \vdash \neg(\neg A)}{\Gamma \vdash A} \mathrm{DNE}$$

Law of Excluded Middle (LEM):

$$\vdash A \vee \neg A$$

Rules for Quantifiers

- The following rules can be thought of as counterparts of those for $\wedge$ and $\vee$ where, instead of just two "juncts", there is one for each individual in the domain of quantification.
- These rules can be added to either IPL or CPL to obtain either intuitionistic or classical versions of FOL.

Rules for the Universal Quantifier

$\forall$-Elimination, or Universal Instantiation (UI):

$$\frac{\Gamma \vdash \forall x A}{\Gamma \vdash A[x/t]} \; \forall\mathrm{E}$$

Note: here '$A[x/t]$' is the formula resulting from replacing all free occurrences of x in A by the term t. This is only permitted if t is "free for x in A", i.e. the replacement does not cause any of the free variables of t to become bound.

$\forall$-Introduction, or Universal Generalization (UG)

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x A} \; \forall\mathrm{I}$$

Note: here the variable x is not permitted to be free in any of the hypotheses in Γ.

Rules for the Existential Quantifier

$\exists$-Elimination:

$$\frac{\Gamma \vdash \exists x A \qquad \Delta, A[x/y] \vdash C}{\Gamma, \Delta \vdash C} \; \exists\mathrm{E}$$

Note: here y must be free for x in A and not free in A.

$\exists$-Introduction, or Existential Generalization (EG):

$$\frac{\Gamma \vdash A[x/t]}{\Gamma \vdash \exists x A} \; \exists\mathrm{I}$$

Note: here t must be free for x in A.