Introduction to Higher Order Logic

Carl Pollard

November 29, 2011

Extending TLC to HOL (Church 1940)

- Start with a TLC.
- Add a type t for truth values.
- Add equality symbols $=_A$ for all types.
- Re-express the TLC term equivalences as object-language axioms about equality.
- *Define* the usual (term-level, not type-level) classical logical connectives and quantifiers in terms of λ and equality.

A Logic Defined in this Way:

- has all the term equalities expected in TLC ('lambda conversion')
- has all the (term-level) theorems of classical FOL
- allows quantification over variables of all types.

Historical Synopsis of Classical HOL

- Henkin(1947/1950) reaxiomatized Church's (1940) Simple Theory of Types
  - added a key axiom (*Truth-Value Extensionality*) identifying identity of truth values with bi-implication, and
  - proved completeness relative to the class of set-theoretic models that bear his name.
- Gallin (1975) showed that Henkin's HOL with two basic types (besides t) instead of just one was equivalent to Montague's IL
- Groenendijk and Stokhof (1980s) started using Ty2 instead of IL for NL semantics.
- Lambek and Scott (1986) added *subtyping* (analogous to the Axiom of Separation in set theory), and allowed a wider class of (not necessarily set-theoretic) models (*toposes*).

Our HOL

- It has everything positive TLC has (including the type constructors T (interpreted as a singleton) and $A \wedge B$ (interpreted as cartesian product)
- Our $A \rightarrow B$ is what Montague wrote as $\langle A, B\rangle$.
- We will have **subtyping** (Lambek and Scott 1986), to be described soon.

HOL: a Closer Look

1. We start with a positive TLC and add a new type t. (This is part of the logic, not a basic type added at the user's discretion.)
2. Terms of type t are called **formulas**. (Note: 'formula' is now ambiguous between 'type' and 'term of type t'.)
3. Axioms will ensure that $I(\mathrm{t})$ has exactly two members (called **truth values**), for any interpretation I.
4. For each type A, we have a constant $=_A : (A \wedge A) \rightarrow \mathrm{t}$, written infix ($a = b$). The type subscript is usually omitted.
5. $I(=_A)$ is the identity relation on $I(A)$.

Classical Connectives and Quantifiers are Definable

Here ϕ is a metavariable over formulas, x is a variable of type A, and s, t are variables of type t:

1. $\mathsf{true} =_{\mathrm{def}} * = *$
2. $\forall_x.\phi =_{\mathrm{def}} \lambda_x.\phi = \lambda_x.\mathsf{true}$
3. $\mathsf{false} =_{\mathrm{def}} \forall_t.t$
4. $\phi \wedge \psi =_{\mathrm{def}} (\phi, \psi) = (\mathsf{true}, \mathsf{true})$
5. $\phi \rightarrow \psi =_{\mathrm{def}} \phi = (\phi \wedge \psi)$
6. $\phi \leftrightarrow \psi =_{\mathrm{def}} (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$
7. $\neg\phi =_{\mathrm{def}} \phi \rightarrow \mathsf{false}$
8. $\phi \vee \psi =_{\mathrm{def}} \neg[(\neg\phi) \wedge (\neg\psi)]$
9. $\exists_x.\phi =_{\mathrm{def}} \neg\forall_x.\neg\phi$

Numerous Options for Axiomatizing HOL

- Gallin (Ty2, 1975) (essentially follows Henkin 1950);
- Carpenter (1997) (essentially follows Andrews 1986);
- Lambek and Scott (1986) have subtyping (see below), and the option of 'going intuitionistic' (dropping Excluded Middle from the term logic)

- We'll remain agnostic about how to axiomatize HOL, and just mention some useful rules and axioms (or theorems, depending on the choice of axiomatization).
- We write $\vdash \phi$ to mean 'ϕ is provable in HOL'. (Note that '$\vdash$' is the same symbol used in typing judgments.)

Equality is an Equivalence Relation

1. $\vdash a = a$ (reflexivity)
2. if $\vdash a = b$, then $\vdash b = a$ (symmetry)
3. If $\vdash a = b$ and $\vdash b = c$, then $\vdash a = c$ (transitivity)

Rules for Substitution of Equals

1. if $\vdash a = c$ and $\vdash b = d$, then $\vdash (a, b) = (c, d)$
2. if $\vdash f = g$ and $\vdash a = b$, then $\vdash f(a) = g(b)$
3. if $\vdash a = b$, then $\vdash \lambda_x.a = \lambda_x.b$

Axioms for Cartesian Products

1. if $\vdash a : \mathrm{T}$, then $\vdash a = *$
2. $\vdash \pi(a, b) = a$
3. $\vdash \pi'(a, b) = b$
4. $\vdash (\pi(c), \pi'(c)) = c$

Axioms for Lambda Conversion

1. $\vdash \lambda_{x \in A}.b = \lambda_{y \in A}.[y/x]b$ (rule α)
2. $\vdash ((\lambda_{x \in A}.b)\ a) = [a/x]b$ (rule β)
3. if $\vdash f : A \to B$ and x is not free in f, then
   $\vdash (\lambda_{x \in A}.f\ x) = f$ (rule η)

Axiom of Excluded Middle

$\vdash \forall_t.t\ \vee \neg t$

Axiom of Nondegeneracy

$\vdash \neg(\mathsf{true} = \mathsf{false})$

Axioms for Equality of Truth Values

1. $\vdash \phi = (\phi = \mathsf{true})$
2. If $\vdash \phi$ and $\vdash \phi = \psi$, then $\vdash \psi$
3. $\vdash \phi$ iff $\vdash \phi = \mathsf{true}$
4. $\vdash \forall_{s,t}.(s \leftrightarrow t) \rightarrow (s = t)$ (Truth-Value Extensionality)

Motivation for Subtypes

- Standard HOL has no way to say A is a *subtype* of B.
- In an intepretation I, this should mean $I(A) \subseteq I(B)$.
- Example: we will want to *define* the type w (worlds) as a certain subtype of the type p $\rightarrow$ t of sets of propositions (namely the ones which are maximal consistent).

Subtypes (after Lambek and Scott 1986)

If A is a type and a an A-predicate (i.e. a closed term of type $A \rightarrow \mathrm{t}$), then

- A_a is a type
- embed_a is a term of type $A_a \rightarrow A$; and
- Axioms:

$$\vdash \forall_{y,z \in A_a}.((\mathsf{embed}_a\ y) = (\mathsf{embed}_a\ z)) \rightarrow y = z)$$
$$\vdash \forall_{x \in A}.(a\ x) \leftrightarrow \exists_{y \in A_a}.x = (\mathsf{embed}_a\ y)$$

What Subtypes Mean in an Interpretation I

- $I(a)$ is a function from $I(A)$ to truth values
- $I(\mathsf{embed}_a)$ is a one-to-one function from $I(A_a)$ to $I(A)$
- the members of $I(A)$ that $I(a)$ maps to $I(\mathsf{true})$ are the ones that are embedded images of members of $I(A_a)$.

So $I(\mathsf{embed}_a)$ is the function that embeds into $I(A)$ the subset whose characteristic function is $I(a)$.