# Functions

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### Functions

- A relation $F$ between $A$ and $B$ is called a **(total) function** from $A$ to $B$ iff for every $x \in A$, there exists a unique $y \in B$ such that $x\ F\ y$.
- In that case we write $F\colon A \to B$.
- This is often expressed by saying that $F$ **takes** members of $A$ as **arguments** and **returns** members of $B$ as **values** (or, alternatively, **takes its values** in $B$).
- Clearly, $\mathsf{dom}(F) = A$.
- For each $a \in \mathsf{dom}(F)$, the unique $b$ such that $a\ F\ b$ is called the **value** of $F$ **at** $a$, written $F(a)$.
- Alternatively, we say $F$ **maps** $a$ to $b$, written $F\colon a \mapsto b$.
- It is easy to prove that there is a unique set, called $B^A$, whose members are the functions from $A$ to $B$.

### Basic Definitions for Functions

Suppose $F\colon A \to B$, $A' \subseteq A$, and $B' \subseteq B$. Then

- the **restriction** of $F$ to $A'$ is the function from $A'$ to $B$ given by

$$F \upharpoonright A' = \{\langle u, v\rangle \in F \mid u \in A'\}$$

- The **image** of $A'$ by $F$ is the set

$$F[A'] =_{\text{def}} \{y \in B \mid \exists x \in A'(y = F(x))\}$$

- The **preimage** (or **inverse image**) of $B'$ by $F$ is the set

$$F^{-1}[B'] =_{\text{def}} \{x \in A \mid \exists y \in B'(y = F(x))\}$$

  This is more simply described as
  $\{x \in A \mid F(x) \in B'\}$

**Operations**

For any $n \geq 0$, an $n$-ary **(total) operation** on $A$ is a function from $A^{(n)}$ to $A$.

- Many of the useful operations we will encounter are binary operations, i.e. functions from $A \times A$ to $A$.
- A unary operation on $A$ is just a function from $A$ to $A$.
- A nullary operation on $A$ is a function from 1 to $A$.

**Identity Functions**

For any $A$, the identity relation $\mathsf{id}_A$ is a unary operation on $A$, such that, for any $x \in A$,

$$\mathsf{id}_A(x) = x$$

**Function Composition**

- If $F\colon A \to B$ and $G\colon B \to C$, then their (relational) composition $G \circ F$ is a function from $A$ to $C$, namely

$$G \circ F = \{\langle x, z\rangle \in A \times C \mid \exists y \in B(y = F(x) \wedge z = G(y))\}.$$

- For each $x \in A$,

$$G \circ F(x) = G(F(x))$$

**Some Workhorse Functions (1/2)**

Here $n$ is any natural number. The set $2 = \wp(1) = \{0, 1\}$ is often called the set of **truth values**.

- There is a unique function from $A$ to 1, called $\Box_A$.
- There is a unique function from 0 to $A$, called $\Diamond_A$.
- The **successor** function **suc** is the unary operation on $\omega$ that maps each natural number to its successor.
- **Arithmetic** functions such as **addition** $(+)$, **multiplication** $(\cdot)$, and **exponentiation** $(\star)$, are binary operations on $\omega$.

  Soon we'll show how these are defined *recursively*, but first we will need to introduce the Recursion Theorem (RT).

**More Workhorse Functions (2/2)**

- For each function $F\colon A \to 2$, the **kernel** of $F$ is the subset

$$\ker(f) =_{\text{def}} \{x \in A \mid f(x) = 1\}$$

  and for each $B \in \wp(A)$, the **characteristic function of** $B$ **in** $A$ is the function that maps each $x \in A$ to 1 if $x \in B$, and to 0 if $x \in A \setminus B$.

- The members of $A^n$ are called $A$**-strings of length** $n$. These are indispensible for formalizing theories of phonology and syntax.

- Operations on 2 are called **truth functions**. These are used to define the meanings of the FOL logical connectives such as $\neg$, $\wedge$, $\vee$, and $\rightarrow$. and in defining the references of linguistic expressions.

- For any set $A$, we can define on $\wp(A)$ the unary operation of **complement**, and the binary operations of **union**, **intersection**, and **relative complement** (exercise).

**Kinds of Functions**

Suppose $F\colon A \to B$. Then $F$ is called:

- **injective**, or **one-to-one**, or an **injection**, if it maps distinct members of $A$ to distinct members of $B$

- **surjective**, or **onto**, or a **surjection**, if $\mathsf{ran}(F) = B$

- **bijective**, or **one-to-one and onto**, or a **bijection**, or a **one-to-one correspondence**, if it is both injective and surjective.

  *Note:* For any bijection $F\colon A \to B$, we can show that the inverse relation $F^{-1}$ is also a function, and in fact a bijection, from $B$ to $A$.

- A relation $F$ between $A$ and $B$ is called a **partial function** from $A$ to $B$ provided there is a subset $A' \subseteq A$ such that $F$ is a (total) function from $A'$ to $B$.

**Examples of Injective Functions**

- for $A \subseteq B$, the function $\mu_{A,B} : A \to B$ that maps each member of $A$ to itself, called the **embedding** of $A$ into $B$

- the functions $\iota_1$ and $\iota_2$, called **canonical injections**, from the cofactors $A$ and $B$ of a cartesian coproduct $A + B$ into the coproduct, defined by $\iota_1(a) = \langle 0, a\rangle$ and $\iota_2(b) = \langle 1, b\rangle$ for all $a \in A$ and $b \in B$

**Examples of Surjective Functions**

- The **projections** $\pi_1$ and $\pi_2$ of a cartesian product $A \times B$ onto its factors $A$ and $B$ respectively, defined by $\pi_1(\langle a, b\rangle) = a$ and $\pi_2(\langle a, b\rangle) = b$ for all $a \in A$ and $b \in B$.

- Given a set $A$ with an equivalence relation $\equiv$, the function from $A$ to $A/\equiv$ that maps each member of $A$ to its equivalence class

**Examples of Bijective Functions**

- any identity function

- We can prove that **suc** is a bijection from $\omega$ to the set $\omega \setminus \{0\}$ of positive natural numbers.

- For any $A$, there is a bijection from $\wp(A)$ to $2^A$ that maps each subset of $A$ to its characteristic function.

  The inverse of this bijection maps each characteristic function to its kernel.

- The truth function that maps 0 and 1 to each other

- The complement operation on a powerset

- For any set $A$, there is a bijection from $A$ to the set $A^1$ that maps each $a \in A$ to the nullary operation that maps 0 to $a$.

- More generally, for any $n \in \omega$, there is a bijection from $A^{(n)}$ to $A^n$ that maps each $A$-string of length $n$ to an $n$-tuple of elements of $A$.

**Propositional Functions**

- In linguistic semantics, operations on the set P of propositions are used to define the senses of 'logic words' such as *and*, *implies*, and *it is not the case that*.

- More generally, 'genuine' relations (such as loving, owning, being at, and knowing that), as opposed to mathematical relations, are modelled as functions whose values are propositions.

**Modelling Word Senses with Propositional Functions**

- Recall that in our foundations for linguistic semantics, we have assumed that we have a set P of **propositions**, a set W of **worlds**, and a (mathematical) relation @ between propositions and worlds.

- We now assume additionally that we have a set I of **individuals**.

- We then model the sense of 'relational' expressions (such as verbs, predicate adjectives, common nouns, and determiners) by functions from a cartesian product $A_1 \times \ldots \times A_n$ to P, where $n > 0$ is the number of arguments and the choices of the $A_i$ depend on what kinds of things (individuals, propositions, functions from individuals to propositions, etc.) are being related.