Problem Set Four: Functions and Recursive Definition

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Problem 1

For any set U, $\approx_U$ is the binary relation on $\wp(U)$ such that $A \approx_U B$ iff there is a bijection from A to B. Prove that $\approx_u$ is an equivalence relation.

Problem 2

Background

Suppose $\sqsubseteq$ is a preorder on a set A. Then a binary operation $\sqcap$ ($\sqcup$) on A is called a **meet** (**join**) provided, for all $a, b \in A$, $a \sqcap b$ ($a \sqcup b$) is a glb (lub) of the set $\{a, b\}$.

If A is a set with a preorder $\sqsubseteq$ and a meet $\sqcap$, then a binary operation $\dashv$ on A is called a **relative pseudocomplement (rpc)** operation iff for all $a, b, c \in A$:

$$a \sqcap b \sqsubseteq c \text{ iff } a \sqsubseteq b \dashv c$$

that is, $b \dashv c$ is a greatest member of $\{a \in A \mid a \sqcap b \sqsubseteq c\}$.

If A is a set with a preorder $\sqsubseteq$, a bottom $\bot$, a meet $\sqcap$, and an rpc operation $\dashv$, a unary operation $'$ (written as a right superscript) is called a **pseudocomplement** operation iff, for every $a \in A$.

$$a' \equiv a \dashv \bot$$

where $\equiv$ is the equivalence relation induced by the preorder. And finally, a pseudocomplement operation $'$ is called a **complement** operation provided, for all $a \in A$

$$(a')' \equiv a.$$

Now let U be a set, and A its powerset, and let $\sqsubseteq$ be the subset inclusion order $\subseteq_U$.

a. A has a top. What is it?

b. A has a bottom. What is it?

c. A has a meet operation. What is it?

d. A has a join operation. What is it?

e. A has an rpc operation. What is it?

f. A has a pseudocomplement operation. What is it?

h. Show that the pseudocomplement operation on A is a complement operation.

Problem 3

Referring to Problem 2, now suppose U is the set of (Kripke-style) worlds, so that A is the set of propositions and $\sqsubseteq$ is entailment. Remember that in this setting, $p@w$ ('p is true at w') means $w \in p$.

a. What kinds of propositions are tops? How many are there?

b. What kinds of propositions are bottoms? How many are there?

c. In order-theoretic terms, how would you describe the contingent propositions?

d. In order-theoretic terms, how would you describe the possibilities?

e. Thinking of the meanings of the Mathese 'logic words' as operations on propositions, which word corresponds to $\sqcap$? Why?

g. Same question for $\sqcup$.

h. Same question for $\dashv$.

i. Same question for $\not$.

Problem 4

Use induction and the definition of $+$ to prove that for every $n \in \omega$, $\mathbf{suc}(n) = 1 + n$.

Problem 5

Use induction and the definition of $\cdot$ to prove that for every $n \in \omega$, $1 \cdot n = n$.

Problem 6

Use RT to recursively define the **exponentiation** operation $\star$, where $m \star n$ is the natural number customarily written m^n. [Hint: as with $+$ and $\cdot$, start by holding m fixed. The heart of the problem is to correctly identify the appropriate values of X, x, and F to use in applying RT.]

Problem 7

Use RT to give a correct recursive definition of the function h used in the definition of transitive closure.