Problem Set Three: Relations

Carl Pollard
The Ohio State University

October 12, 2011

Problem 1

Let A be any set. In this problem "relation" means "binary relation on A." Prove that:

a. The intersection of two transitive relations is a transitive relation.

b. The intersection of two symmetric relations is a symmetric relation,

c. The intersection of two reflexive relations is a reflexive relation.

d. The intersection of two equivalence relations is an equivalence relation.

Problem 2

Background. For any binary relation R on a set A, the **symmetric interior** of R, written $\mathbf{Sym}(R)$, is defined to be the relation $R \cap R^{-1}$ on A. For example, if R is the relation that holds between a pair of people when the first respects the other, then $\mathbf{Sym}(R)$ is the relation of mutual respect. Another example: if R is the entailment relation on propositions, then the symmetric interior is truth-conditional equivalence.

Prove that the symmetric interior of a preorder is an equivalence relation.

Problem 3

Background. If $\sqsubseteq$ is a preorder, then $\mathbf{Sym}(\sqsubseteq)$ is called the equivalence relation **induced by** $\sqsubseteq$ and written $\equiv_{\sqsubseteq}$, or just $\equiv$ if it's clear from the context which preorder is under discussion. If $a \equiv b$, then we say a and b are **tied** with respect to the preorder $\sqsubseteq$.

Also, for any relation R, there is a corresponding asymmetric relation called the **asymmetric interior** of R, written $\mathbf{Asym}(R)$ and defined to be

$R \setminus R^{-1}$. For example, the asymmetric interior of the love relation on people is the unrequited love relation.

In a context where there is a fixed preorder $\sqsubseteq$, recall that $a \sqsubseteq b$ is usually read "a is less than or equivalent to b"; if in addition $\sqsubseteq$ is antisymmetric (i.e. an order), then it is read "a is less than or equal to b" because the only thing tied with a is a itself.

In a context where there is a fixed preorder $\sqsubseteq$, $\mathbf{Asym}(\sqsubseteq)$ is abbreviated $\sqsubset$, read "strictly less than". [Confession: the symbol I really wanted here was the squared-off version of $\subsetneq$, but I couldn't figure out how to produce it!] Careful: if $a \sqsubset b$, then not only are a and b not equal, but also they are not equivalent.

If $\sqsubseteq$ is a preorder, then we say c is **strictly between** a and b to mean that $a \sqsubset c$ and $c \sqsubset b$.

Given a preorder $\sqsubseteq$ on a set A and $a, b \in A$, we say a **is covered by** b if $a \sqsubset b$ and there is nothing strictly between them. The relation consisting of all such pairs $\langle a, b \rangle$ is called the **covering relation induced by** $\sqsubseteq$ and written $\prec_{\sqsubseteq}$, or just $\prec$ when no confusion can arise.

a. Prove that $\prec$ is an intransitive relation.

b. Let $\leq$ be the usual order on ω. What is the induced covering relation? [Hint: it is a function that we have already encountered.]

c. Remember from grade school (or maybe middle school?) that any positive rational number less than 1 can be represented in a unique way by a fraction m/n where both m and n are nonzero natural numbers, $m < n$, and m and n have no common factor (other than 1), i.e. the fraction is 'reduced to lowest terms'. In this context, let $\leq$ represent the usual order on the set of such numbers (just take it on faith that there *is* such a set). What is the induced covering relation on $\leq$?

d. Let U be a set, $\subseteq_U$ the subset inclusion relation on $\wp(U)$, and $\prec$ the corresponding covering relation. In simple English, how do you tell by looking at two subsets A and B of U whether $A \prec B$?

Problem 4

Background. For any binary relation R on A, the **reflexive closure** of R, written $\mathbf{Refl}(R)$, is defined to be the relation $R \cup \mathsf{id}_A$. Clearly if R is transitive then $\mathbf{Refl}(R)$ is a preorder.

Now suppose P is the set of all people who have ever lived (i.e. a set that we are using to *represent* the collection of people who have ever lived) and let

D be a transitive asymmetric relation on P used to represent the relation that holds between a pair of people if the first is a descendant of the second. Let $\sqsubseteq =_{\text{def}} \mathbf{Refl}(D)$, and $\prec$ the corresponding covering relation. To keep things simple, assume (counterfactually, of course) that (1) every person has exactly two parents, and (2) any two people with a parent in common have both of their parents in common.

a. In plain English, why did we require that D be transitive and asymmetric? (That is, what facts of life are modelled by imposing these conditions on D?)

b. Write a formula (sentence made up of Mathese symbols) expressing the condition (1). [Hint: it is much easier to express this in terms of $\prec$ than in terms of D!]

c. Write a formula expressing the condition (2). [Same hint as immediately above.]

d. Suppose a and b are two people. Write a formula that means that a and b are cousins. (To eliminate any variation in or unclarity about the meaning of English kin terms, assume that a person's cousins are the children of his or her parents' siblings, not counting ones with whom he or she has a parent in common).

Translate the following formulas into plain English, using familiar kinship terms.

e. $a \prec b$

f. $b \prec^{-1} a$

g. $a \prec \circ \prec b$

h. $a\ (\prec \circ \prec^{-1}) \setminus \mathsf{id}_P\ b$

i. $a\ (\prec^{-1} \circ \prec) \setminus \mathsf{id}_P\ b$