Proving Equivalence

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Equivalences and Implication

- We saw recently that it's always possible (for any two PL propositions φ and ψ) to put either one on either side of a conjunction.
- Another way of saying this is that any time $\varphi \wedge \psi$ is true $\psi \wedge \varphi$ is also true (as our truth tables can verify).
- So we can start from either $\varphi \wedge \psi$ or $\psi \wedge \varphi$ and prove the other.
- Also, now that we have Implication Introduction (→I), we can capture a piece of the entailment present in any given proof (Figure 1 shows an example of this).

$$\dfrac{\dfrac{\dfrac{\varphi \wedge \psi \vdash \varphi \wedge \psi}{\varphi \wedge \psi \vdash \psi}\,(\wedge E_2) \qquad \dfrac{\varphi \wedge \psi \vdash \varphi \wedge \psi}{\varphi \wedge \psi \vdash \varphi}\,(\wedge E_1)}{\varphi \wedge \psi \vdash \psi \wedge \varphi}\,(\wedge I)}{\vdash (\varphi \wedge \psi) \to (\psi \wedge \varphi)}\,(\to I)$$

Figure 1: Proof of $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$.

- So, as Figure 1 shows, introducing an instance of the connective → gives us a way to say in the logic that some premise leads to some conclusion.
- But notice that we'd ideally like to make a stronger claim than just $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$.
- That is, we want to be able to say not just that "starting from $\varphi \wedge \psi$, you can deduce $\psi \wedge \varphi$".
- We'd like to have our logic be capable of deriving the fact that $\varphi \wedge \psi$ and $\psi \wedge \varphi$ are *equivalent* statements.
- Remembering that we already have a way to state equivalence in our logic via the biimplicational connective ↔, we add more logical rules.

Inference Rule 9 (Biimplication Introduction)**.**

$$\dfrac{\Gamma \vdash \varphi \to \psi \qquad \Delta \vdash \psi \to \varphi}{\Gamma, \Delta \vdash \varphi \leftrightarrow \psi}\,(\leftrightarrow I)$$

- With Rule 9, it's easy to see why the symbol $\leftrightarrow$ was chosen to represent biimplication.
- It's because a biimplication essentially says "with either side (the antecedent) being true, you get the other side (the consequent) being true."
- The reason biimplication is used to capture equivalence, as our truth tables say, is that if one is true (false) then the other is also true (false).
- There are also elimination rules for $\leftrightarrow$ that let us use equivalences in proofs.

Inference Rule 10 (Biimplication Elimination 1)**.**

$$\frac{\Gamma \vdash \varphi \leftrightarrow \psi}{\Gamma \vdash \varphi \to \psi}\ (\leftrightarrow\mathrm{E}_1)$$

Inference Rule 11 (Biimplication Elimination 2)**.**

$$\frac{\Gamma \vdash \varphi \leftrightarrow \psi}{\Gamma \vdash \psi \to \varphi}\ (\leftrightarrow\mathrm{E}_2)$$

- Now we can actually *prove* that $\varphi \land \psi$ is equivalent to $\psi \land \varphi$ (Figure 2 gives this proof).

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \land \psi \vdash \varphi \land \psi}{\varphi \land \psi \vdash \psi}\ (\land\mathrm{E}_2) \qquad \dfrac{\varphi \land \psi \vdash \varphi \land \psi}{\varphi \land \psi \vdash \varphi}\ (\land\mathrm{E}_1)}{\varphi \land \psi \vdash \psi \land \varphi}\ (\land\mathrm{I})}{\vdash (\varphi \land \psi) \to (\psi \land \varphi)}\ (\to\mathrm{I}) \qquad \dfrac{\dfrac{\dfrac{\psi \land \varphi \vdash \psi \land \varphi}{\psi \land \varphi \vdash \varphi}\ (\land\mathrm{E}_2) \qquad \dfrac{\psi \land \varphi \vdash \psi \land \varphi}{\psi \land \varphi \vdash \psi}\ (\land\mathrm{E}_1)}{\psi \land \varphi \vdash \varphi \land \psi}\ (\land\mathrm{I})}{\vdash (\psi \land \varphi) \to (\varphi \land \psi)}\ (\to\mathrm{I})}{\vdash (\varphi \land \psi) \leftrightarrow (\psi \land \varphi)}\ (\leftrightarrow\mathrm{I})$$

Figure 2: Proof of $(\varphi \land \psi) \leftrightarrow (\psi \land \varphi)$.

- Notice that, in the proof given in Figure 2, there are no premises left of the turnstile.
- This means that what we've proved, namely that $\varphi \land \psi$ and $\psi \land \varphi$ are equivalent to one another, is not contingent on any other assumptions. This is exactly what we want our logic to say about equivalences.
- One technical note: I have used φ and ψ in the proof in Figure 2. While I could have chosen A and B, or any other two propositional letters, I chose to use meta-variables because it shows that this equivalence holds for *any* two propositions φ and ψ (not just atomic or complex ones).

Homework

Problem 1. Starting with the assumptions $A \leftrightarrow B$, $(B \land A) \to C$, and A, give a sequent-style natural deduction proof of $A \to C$.