Interpreting Propositional Logic (Part 2)

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More Truth Tables

Conjunction ($\wedge$)

- Like all the connectives besides negation, conjunction is a **binary connective** because it combines two propositions.
- In a PL sentence of the form $\varphi \wedge \psi$, we call φ and ψ the **left** and **right conjunct**, respectively.
- A PL sentence of the form $\varphi \wedge \psi$ is true under truth assignment if (and only if) both φ and ψ are true under that same assignment. In any other case (i.e., either φ or ψ is false, or both, under that assignment), $\varphi \wedge \psi$ is false. The truth table for conjunction (Table 1) reflects this.

φ	ψ	$\varphi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

Table 1: Truth table for conjunction.

- Notice that, like the truth table for negation, the truth table for conjunction applies to both atomic and complex propositions.
- Logical conjunction is used to model English conjunctions *and, but, however*, etc.:

  (1) a. Clint lives in Lake Wobegon.
  b. Clarence lives in Lake Wobegon.
  c. Clint and Clarence live in Lake Wobegon.

  Letting A be the proposition expressed by (1a) and B the one expressed by (1b), we see that $A \wedge B$, the proposition expressed by (1c), can only be true if both A and B are true. So the truth table in Table 1 corresponds with our intuitions about English *and*.

Disjunction ($\vee$)

- PL sentences of the form $A \vee B$ are called **disjunctions**; A and B are called the **left** and **right disjuncts**, respectively.

- The binary connective $\vee$ corresponds to **some** uses of the English word *or*, but probably not most.

- For various reasons, the *or* denoted by $\vee$ is **inclusive** or, whereas the *or* often used in NL is an **exclusive** *or*. The differences:

  Exclusive disjunctions are true if one or other of the disjuncts is true, but crucially not true if both are true.

  Inclusive disjunctions are true in the same cases where exclusive disjunctions are true with the exception that they are also true when both disjuncts are true.

  So exclusive disjunction corresponds to English sentences like *either A or B, but not both*, while inclusive disjunction corresponds to English sentences like *either A or B or possibly both*.

- Given that $\vee$ is inclusive, a PL sentence of the form $\varphi \vee \psi$ should be intepreted as true in any of the following cases:

  - φ is true,
  - ψ is true, or
  - both φ and ψ are true.

  And $\varphi \vee \psi$ should *only* be false when both φ and ψ are false. The truth table in Table 2 summarizes this.

φ	ψ	$\varphi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

Table 2: Truth table for disjunction.

- For one possible explanation for why NL *or* is often exclusive, notice that whenever a PL sentence like $\varphi \wedge \psi$ is true, then necessarily $\varphi \vee \psi$ is true (because both φ and ψ are true). Now consider the following:

  (2) a. Myrtle is at the Chatterbox Cafe.
  b. Pastor Ingqvist is at the Chatterbox Cafe.
  c. Either Myrtle or Pastor Ingqvist is at the Chatterbox Cafe.
  d. Myrtle and Pastor Ingqvist are both at the Chatterbox Cafe.

Now imagine a situation where two people are having a conversation and one of them knows that both (2a) and (2b) are true (*not* just one or the other). Then it would seem somewhat uncooperative to say (2c) in this context because the use of *or* communicates that the speaker doesn't know whether both are true. Instead, a cooperative conversation partner would probably use (2d).

- On the other hand, NL *or* is clearly inclusive sometimes:

  (3) a. Does Pastor Ingqvist drink or smoke?
  b. No, he doesn't.

  The answer negates the disjunction contained in the question. But how should we interpret it? If *or* is taken to be exclusive, then either Pastor Ingqvist doesn't smoke or he doesn't drink, or he does both! This is clearly not what is intuitively meant by the negative answer in (3b), so there must be some uses of *or* that are inclusive.

Implication ($\rightarrow$)

- A PL sentence like $\varphi \rightarrow \psi$ is taken to be

  true whenever φ is false or ψ is true, and

  false if φ is true but ψ is false.

φ	ψ	$\varphi \rightarrow \psi$
T	T	T
T	F	F
F	T	T
F	F	T

Table 3: Truth table for implication.

- In a sentence of the form $\varphi \rightarrow \psi$, φ is called the **antecedent** and ψ is called the **consequent**.
- Implication in PL corresponds to the English words *if ... (then) ...*, *given that ...*, *assuming that ...*, *provided that ...*, *only if ...*, etc.
- Consider the following propositionally equivalent English sentences:

  (4) a. If they have ketchup on them, Clint likes Powdermilk biscuits.
  b. Clint likes Powdermilk biscuits if they have ketchup on them.

  Letting K be the proposition expressed by *They have ketchup on them* and L be the proposition denoted by *Clint likes Powdermilk biscuits*, we can translate both sentences in (4) into PL as $(K \rightarrow L)$.

- So implication in PL essentially expresses a link between two propositions, or a commitment to the truth of a certain proposition (the consequent) given the truth of another proposition (the antecedent).

- Notice that the antecedent being false means that an implication is vacuously true. This is like writing someone a check that they never cash–the money was still in the bank.
- Alternatively, we can think of the truth of the antecedent as a **sufficient condition** for the truth of the consequent, and the truth of the consequent as a **necessary condition** for the truth of the antecedent.
- It is also important to note that implication only expresses a link between the truth of two propositions, not causality. Also, the *if ... then ...* construction often used in natural languages (like *and*) often expresses something extra that we're not considering here: temporal precedence.

Biimplication ($\leftrightarrow$)

- PL sentences of the form $\varphi \leftrightarrow \psi$ are interpreted as

  true if both φ and ψ are true or both φ and ψ are false, and

  false if one of φ or ψ is true (false) but the other is false (true).

φ	ψ	$\varphi \leftrightarrow \psi$
T	T	T
T	F	F
F	T	F
F	F	T

Table 4: Truth table for biimplication.

- Notice that $\varphi \leftrightarrow \psi$ means the same thing as $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. This is the reason the symbol $\leftrightarrow$ was chosen to represent biimplication.
- So sentences like $\varphi \leftrightarrow \psi$ express a kind of double promise. The truth of either φ or ψ implies the truth of the other.
- Notice also that $\varphi \leftrightarrow \psi$ expresses a stronger claim than just $\varphi \rightarrow \psi$. In a PL sentence like $\varphi \leftrightarrow \psi$, both φ and ψ are a necessary **and** sufficient condition for the other.
- The English constructions *... if and only if ...* and *just in case* are taken to correspond to biiimplication. But these constructions are usually used in technical settings, not very often in colloquial ones:

(5) a. An argument is a set of declarative sentences, some of which are premises and some of which are conclusions.

b. Something counts as an argument if and only if it is a set of declarative sentences, some of which are premises and others conclusions.

While both sentences in (5) express the same proposition, the one in (5b) makes it more clear that we're talking about a biimplication. Another example:

(6) a. A conjunction is true just in case both conjuncts are true.
b. A disjunction is true if both disjuncts are true.

Notice that (6a) makes it clear that a conjunction is true when both conjuncts are true and **only** in that case (a stronger claim than just using *if*). Now consider what would happen if we replaced the *if* in (6b) with *just in case* or *if and only if*.

Homework

Problem 1. Recalling the discussion of necessary and sufficient conditions, how would you translate the sentences in (7) into PL?

(7) a. Buying bait is a necessary condition for Clint to go fishing.
b. Stopping off at the Sidetrack Tap is not necessary for Clarence to be productive at work.
c. Eating ketchup is not sufficient for Myrtle to be happy.
d. To get coasters, it is sufficient to go by Skogelin's 5 and dime.

Problem 2. How do the propositions expressed by the three sentences in (8) differ? How would you translate each of them into PL?

(8) a. Clarence catches a Walleye if he chooses the right bait.
b. Clarence catches a Walleye only if he chooses the right bait.
c. Clarence catches a Walleye if and only if he chooses the right bait.

Problem 3. Can you think of a good paraphrase of the sentence in (9)? How would you translate it into PL?

(9) Wally won't eat Powdermilk biscuits unless Evelyn makes them.