Entailments and Equivalence

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- With natural deduction (ND), we can now draw graphs of arguments and examine them for validity without having to use truth tables.
- But we'd also like to be able to discover equivalences between propositions expressed in propositional logic (PL).
- Also, we'd like to have a way to say (formally) when some premise(s) entail some conclusion.

Proofs of Equivalence

- Since we started talking about $\wedge$ (logical conjunction) and how it conjoins propositions, we've noted that for any propositions φ and ψ the proposition $\varphi \wedge \psi$ is equivalent to the proposition $\psi \wedge \varphi$.
- We've said things like "it doesn't matter which side" or "they're interchangeable" to indicate this is our informal discussions.
- Notice that you *could* use a truth table to quickly convince yourself that $\varphi \wedge \psi$ is true in all the same cases that $\psi \wedge \varphi$ is true.
- But with ND, we don't need to do this anymore. We can give a syntactic proof (shown in Figure 1) that says the same thing without needing to consider every way the world could be.

$$\dfrac{\dfrac{\dfrac{}{\varphi \wedge \psi}\,(\text{Hyp})}{\psi}\,(\wedge\text{E}_2) \qquad \dfrac{\dfrac{}{\varphi \wedge \psi}\,(\text{Hyp})}{\varphi}\,(\wedge\text{E}_1)}{\psi \wedge \varphi}\,(\wedge\text{I})$$

Figure 1: Proof of $\psi \wedge \varphi$ from $\varphi \wedge \psi$.

ND Proofs and Entailment

- In ND proofs, we can see graphically how an argument is laid out: the premises are at the top, the conclusion at the very bottom, and everything in between is an inference step allowed by one of our rules.

- Notice how this mirrors the notion of entailment for deductively valid arguments–starting with true premises, you arrive at a conclusion that must be true (no matter how the world is).
- We can mention this in the meta-language, but sometimes we'd actually like to use the fact that some premise(s) entail some conclusion in yet another proof.
- One way to do this is to simply re-use the conclusion as a new premises (we'll do this a lot).
- Another way is to make an implication using one of the premises as the antecedent and the conclusion as the consequent (Rule 7 formalizes this step).

Inference Rule 7 (Implication Introduction)**.**

$$\begin{array}{c} \dfrac{}{[\varphi]_i}\ (\text{Hyp}) \\ \vdots \\ i\ \dfrac{\psi}{\varphi \to \psi}\ (\to\text{I}) \end{array}$$

- Rule 7 looks complicated (we'll fix this later), but all it says is that if you've got some premise that you've assumed using Hypothesis, you can later **withdraw** or **revoke** that hypothesis and make it the antecedent of a conditional.
- For bookkeeping, we write $[\quad]_i$ around the withdrawn hypothesis, and the same number i to the left of the introduction step.
- Notice that our rules are getting a bit sloppy–now we have to know that $\vdots$ means something like "any number of inference steps". So the rule of Implication Introduction is not as formally rigorous as some of the other rules we've used up to this point because it relies more on the meta-language.
- Figure 2 shows an example of this rule in action.

$$2\ \dfrac{1\ \dfrac{\dfrac{\dfrac{}{[A\to B]_2}\ (\text{Hyp}) \qquad \dfrac{\dfrac{}{[A\wedge C]_1}\ (\text{Hyp})}{A}\ (\wedge\text{E}_1)}{B}\ (\to\text{E})}{(A\wedge C)\to B}\ (\to\text{I})}{(A\to B)\to((A\wedge C)\to B)}\ (\to\text{I})$$

Figure 2: Example proof using Rule 7.

Homework

Problem 1. Give a formal ND proof of $\neg\neg A \to B$ that assumes as premises only $\neg\neg A$ and $A \to B$.