PROBLEM SET EIGHT: LAMBEK CALCULUS

Background

Traditionally, logic was concerned with establishing principles of valid argumentation, or, to put it another way, with determining under what conditions a proposition (the conclusion) can be said to follow from other propositions (the premisses). More specifically, **propositional** logic seeks to establish such principles solely on the basis of how the expressions which express propositions—declarative sentences—are combined, leaving aside how simple sentences themselves are constructed from smaller expressions (such as nouns and verbs).

But from a mathematical point of view, we can identify propositions with elements of a pre-boolean algebra, 'following from' with the preorder (entailment), and the meanings of the various ways of combining sentences with the algebra operations. And so. we can reconceptualize (classical) propositional logic as a system for proving theorems about entailment in a pre-boolean algebra. And, more generally, we can have different *kinds* of propositional logics, each of which is suited to a different class of preordered algebras, e.g. *intuitionistic* logic to pre-heyting algebras, *bilinear* logic (aka Lambek calculus) to residuated pre-semigroups, etc. Once we generalize in this way, we are no longer committed to thinking of the formulas that the logic manipulates as sentence meanings and 'following from' (usually written as $\vdash$, read 'turnstile') as entailment, but different interpretations might be available (depending on the kind of logic). For example, in classical logic, the familiar interpretation of a 'judgment'

$$\Gamma \vdash A$$

where Γ is a set of formulas and A is a formula is that the proposition expressed by A is guaranteed to be true as long as the propositions expressed by the formulas in Γ are. But in Lambek calculus,

$$\Gamma \vdash A$$

is most often interpreted this way: linguistic expressions belonging to the syntactic types listed in Γ can be strung together in the given order to form an expression of syntactic type A. (Note also that, whereas in classical logic, Γ is a *set* of formulas, in Lambek calculus, it is a *string* of formulas, because order and repetitions make a difference.)

1 The (Product-Free) Lambek Calculus

We begin by fixing a set of **atomic formulas** (say, $\{\mathrm{NP}, \mathrm{N}, \mathrm{S}\}$), and then recursively defining the set of **formulas** as follows:

a. An atomic formula is a formula.

b. If A and B are formulas, then so are $A\backslash B$ and A/B.

We then define a **sequent** to be an ordered pair $\langle \Gamma, A \rangle$ where Γ is a non-null string of formulas and A is a formula. Γ is called the **context** of the sequent, the formula occurences in the context are called the **hypotheses**, and A is called the **statement**.

Next, we recursively define a set of sequents called the **derivable** sequents. That is, we recursively define a relation, called **derivability** (written '$\vdash$'), between nonempty strings of formulas and formulas. Usually we read $\Gamma \vdash A$ as 'Γ derives A' rather than '$\langle \Gamma, A \rangle$ is a derivable sequent'. An assertion of this form is called a (derivability) **judgment**.

Here's the definition of the derivability relation (throughout, italic capital roman letters are metavariables that range over formulas, and capital Greek letters are metavariables that range over non-null strings of formulas):

Base clause: Axioms

$$A \vdash A$$

Recursion Clauses (Inference Rules)

\ Elimination:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash A\backslash B}{\Gamma\Delta \vdash B}$$

\ Introduction:

$$\frac{A\Gamma \vdash B}{\Gamma \vdash A\backslash B}$$

/ Elimination:

$$\frac{\Gamma \vdash B/A \qquad \Delta \vdash A}{\Gamma\Delta \vdash B}$$

/ Introduction:

$$\frac{\Gamma A \vdash B}{\Gamma \vdash B/A}$$

In an inference rule, the judgments above the horizontal line are called the **premisses** and the judgment below the line is called the **conclusion**. The whole rule is to be understood as a conditional, with the conjunction of the premisses as antecdent and the conclusion as the consequent: if all premisses are true, then so is the conclusion. (An axiom can be understood as an inference rule with no premisses, in other words just a direct assertion, in this case that (no matter what A is), A derives A.) Also, in an Introduction rule, the hypothesis A which appears in the context of the premiss but not in the context of the conclusion is called the **withdrawn** hypothesis.

And finally, we define a **(formal) proof** of a judgment Σ to be a tree with each node labelled by a judgment, such that the following three conditions hold:

1. The root is labelled by Σ.
2. Each leaf is labelled by an axiom.
3. For each nonleaf x, the label of x is the conclusion of a rule whose premisses are the labels of x's daughters.

It should be intuitively obvious (though we omit the informal metalanguage proof) that a judgment $\Gamma \vdash A$ is true (i.e. Γ does indeed derive A) iff there is a formal proof of it. In that case we call the judgment a **theorem** (of the Lambek calculus).

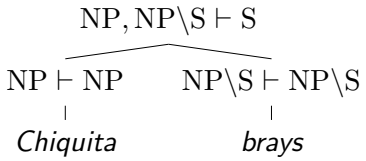
2 Linguistic Application: Lambek Grammars

A **Lambek grammar** uses a Lambek calculus to recursively assign word strings to syntactic types as follows. First (just as sketched in the last class) we have a lexicon that assigns words (more precisely, word strings of length

one) to one or more syntactic types, e.g. Chiquita : NP, brays : NP\S, etc. This is the base of the recursion.

Next, the recursion clause is this: if the strings $s_0, \ldots, s_n$ are assigned to types $A_0, \ldots, A_n$ respectively (for some $n \in \omega$), and $A_0 \ldots A_n \vdash A$, then the string $s_0 \ldots s_n$ is assigned to type A.

If we only use the Elimination rules (which play a role in Lambek calculus analogous nto the role played by $\rightarrow$-Elimination (Modus Ponens) in familiar classical (or intuitionistic) logic, then it is not hard to see that, for a given lexicon, a Lambek grammar is essentially equivalent to an applicative categorial grammar (as described in class), in the sense of producing the same assignment of strings to types. For example, *Chiqita brays* is a sentence because NP NP\S $\vdash$ S is a Lambek theorem (the formal proof uses two axioms and one instance of \E):



(Notes: (1) in proof trees, hypotheses are separated by commas, since otherwise the tree-drawing program jams them together; and (2) as a mnemonic, we write words below the corresponding axiom instances, even though technically they are not part of the formal proof.)

Problem 1

Give a formal proof of the Lambek theorem NP TV NP $\vdash$ S that uses only Elimination rules. (Note: 'TV' (transitive verb) abbreviates (NP\S)/NP.)

However, the presence of the Introduction rules (which are analogous to the classical/intuitionistic $\rightarrow$-Introduction, aka Hypothetical Proof or Conditionalization) changes the nature of the game. For one thing, new proofs become available which don't comport well with the traditional notion of syntactic constituency (as discussed in introductory syntax courses).

Problem 2

Give a different formal proof of NP TV NP $\vdash$ S in which the topmost rule (the one that licenses the root S of the proof tree) is /E. (Hint: the proof tree will have FOUR leaves, not three, with the third one corresponding to a 'hypothetical' NP—analogous to what syntacticans call a 'trace'.)

In the application to linguistics, the way that the Introduction rules are used is as follows. Instead of every leaf in the proof tree corresponding to a word, as before, we now allow some leaves to correspond to 'hypothetical' expressions which are inaudible (that is, they are associated with the null string). And when an Introduction rule is used higher in the proof tree, a hypothesis can be withdrawn only if it originated from a leaf associated with a hypothetical (inaudible) expression. Additionally, a hypothesis can be withdrawn using \E (/E) only if the corresponding leaf is the leftmost (rightmost) leaf of the proof tree.

Problem 3

Prove the Lambek theorem $A/B \;\; B/C \vdash A/C$. (This is known as Right Composition. There's also a left-handed version.)

Problem 4

Prove the Lambek theorem $A \vdash B/(A\backslash B)$. (This is known as Type Raising on the Left. There's also a right-handed version.)

Categorial grammarians often assume that conjunctions (such as *and*, *or*, and *but*) each have an infinite number of syntactic types, each of the form $(A\backslash A)/A$, which is abbreviated $\mathrm{K}(A)$ (here A is a metavariable rsanging over syntactic types). This allows a conjunction to combine first with an A on the right, and then with an A on the left, to produce an A.

Problem 5

Use the preceding idea to give an analysis of the sentence *Juan liked, but Maria hated, Chiquita*. (Examples of this kind exemplify the phenomenon known as 'right node raising') [Hints: (1) The proof tree will have two 'traces' in it. (2) The traces in the proof tree should look like this:

$$\begin{array}{l} \mathrm{NP} \vdash \mathrm{NP} \\ \quad\; | \\ \quad\; e \end{array}$$

where e is the null string.]